SYZYGIES OF SEGRE EMBEDDINGS AND Δ-MODULES

ANDREW SNOWDEN

ABSTRACT. We study syzygies of the Segre embedding of $\mathbf{P}(V_1) \times \cdots \times \mathbf{P}(V_n)$, and prove two finiteness results. First, for fixed p but varying n and V_i, there is a finite list of "master p-syzygies" from which all other p-syzygies can be derived by simple substitutions. Second, we define a power series f_p with coefficients in something like the Schur algebra, which contains essentially all the information of p-syzygies of Segre embeddings (for all n and V_i), and show that it is a rational function. The list of master p-syzygies and the numerator and denominator of f_p can be computed algorithmically (in theory). The central observation of this paper is that by considering all Segre embeddings at once (i.e., letting n and the V_i vary) certain structure on the space of p-syzygies emerges. We formalize this structure in the concept of a Δ-module. Many of our results on syzygies are specializations of general results on Δ-modules that we establish. Our theory also applies to certain other families of varieties, such as tangent and secant varieties of Segre embeddings.

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1. Introduction

Let $V_1, \ldots, V_n$ be finite dimensional complex vector spaces. The Segre embedding is the map

$$\mathbf{P}(V_1) \times \cdots \times \mathbf{P}(V_n) \to \mathbf{P}(V_1 \otimes \cdots \otimes V_n)$$

taking a tuple of lines to their tensor product. Despite its fundamental nature, the syzygies of this embedding are poorly understood (excluding some special cases, such as when $n = 2$). The purpose of this paper is to establish a general structural theory of these syzygies. We show that for p fixed, but n and the V_i variable, the collection of all p-syzygies can be given an algebraic structure, and that this structure has favorable finiteness properties. One consequence of this result is that almost any question about p-syzygies — such as: is the module of 13-syzygies of every Segre embedding generated in degrees at most 20? — can be resolved by a finite computation (in theory). In the rest of the introduction, we describe this structure and present our two main theorems about it. As the precise statements of these theorems are somewhat technical, we begin with an informal discussion that attempts to elucidate the essential content of the first theorem.

1.1. **The first theorem (informal).** The equations of the Segre embedding have been known for as long as the embeddings themselves have been known, and are familiar to most anyone who has studied algebraic geometry. We briefly recall them. First suppose that $n = 2$, i.e., consider the embedding of $\mathbf{P}(V_1) \times \mathbf{P}(V_2)$ into $\mathbf{P}(V_1 \otimes V_2)$. Let $\{x_i\}_{i \in I_1}$ be coordinates on $\mathbf{P}(V_1)$, let $\{x_j\}_{j \in I_2}$ be coordinates on $\mathbf{P}(V_2)$ and let $\{x_{ij}\}_{(i,j) \in I_1 \times I_2}$ be coordinates on $\mathbf{P}(V_1 \otimes V_2)$. Then the equations

$$x_{i_1 j_1} x_{i_2 j_2} - x_{i_1 j_2} x_{i_2 j_1} = 0$$

cut out $\mathbf{P}(V_1) \times \mathbf{P}(V_2)$ inside $\mathbf{P}(V_1 \otimes V_2)$. Now suppose that $n = 3$. Then, in the obvious notation, we have the equation

$$x_{i_1 j_1 k_1} x_{i_2 j_2 k_2} - x_{i_1 j_1 k_2} x_{i_2 j_2 k_1} = 0.$$

This is not symmetrical in the i, j and k indices, and so we get two other families of equations by changing the roles of the indices. The resulting collection of equations cuts out $\mathbf{P}(V_1) \times \mathbf{P}(V_2) \times \mathbf{P}(V_3)$ inside of $\mathbf{P}(V_1 \otimes V_2 \otimes V_3)$. The situation for $n > 3$ is similar.

The equations we have just described are quite similar to each other. Indeed, they can all be recovered from the single "master equation"

$$x_{12} x_{34} - x_{14} x_{32} = 0$$

by certain substitution rules. When $n = 2$ we obtain all the equations by replacing the numbers 1 and 3 (resp. 2 and 4) in the master equation by elements of I_1 (resp. I_2). When $n = 3$ we obtain all the equations by replacing the numbers 1 and 3 by elements of $I_1 \times I_2$ and 2 and 4 by elements of I_3, or by doing this with the roles of I_1, I_2 and I_3 interchanged. For general n, we obtain the equations for $\mathbf{P}(V_1) \times \cdots \times \mathbf{P}(V_n)$ by partitioning the set $\{1, \ldots, n\}$ into two disjoint subsets A and B and then replacing 1 and 3 by elements of $\prod_{k \in A} I_k$ and 2 and 4 by elements of $\prod_{k \in B} I_k$.

We now examine the syzygies between equations (2-syzygies). We again begin with the case $n = 2$. Consider the determinant

$$\begin{vmatrix} x_{i_1 j_1} & x_{i_1 j_2} & x_{i_1 j_3} \\ x_{i_1 j_1} & x_{i_1 j_2} & x_{i_1 j_3} \\ x_{i_2 j_1} & x_{i_2 j_2} & x_{i_2 j_3} \end{vmatrix}$$

The first two rows are repeated, and so the determinant vanishes when considered as a polynomial in the variables x_{ij}. Now, the 2×2 minors along the bottom are exactly the previously constructed equations. Thus by taking the Laplace expansion of the determinant along the top row we obtain a linear relation between equations, i.e., a syzygy. This construction can be modified to produce 2-syzygies for $n > 2$, in the same way that the $n = 2$ equations were modified to get equations for $n > 2$. These syzygies generate the module of 2-syzygies, for any n.

As was the case for equations, all the 2-syzygies just produced can be recovered from a "master syzygy" by certain substitution rules. The master syzygy is just the one obtained from the Laplace expansion of

$$\begin{vmatrix} x_{13} & x_{14} & x_{15} \\ x_{13} & x_{14} & x_{15} \\ x_{23} & x_{24} & x_{25} \end{vmatrix}$$

The substitution rules are similar: partition $\{1, \ldots, n\}$ into two sets A and B and replace 1 and 2 by elements of $\prod_{k \in A} I_k$ and 3, 4 and 5 by elements of $\prod_{k \in B} I_k$.

Given these observations, one might ask: are there master higher syzygies? An affirmative answer is provided by the following result, the first of our two main theorems:

Theorem A (Informal version)**.** *Given p, there is a finite list of "master p-syzygies" from which all p-syzygies can be obtained through the substitution procedures sketched above.*

This result is slightly different than what we have seen for equations and syzygies of equations in that there is no longer a single master syzygy, but rather a finite list of them. The theorem does not provide any canonical list of master syzygies, it just asserts the existence of such a list. The proof of the theorem can easily be adapted to yield an algorithm which, given p, produces a

list of master p-syzygies. Unfortunately, this algorithm is fantastically impractical, even for small p — it involves linear algebra over a polynomial ring in p^{2p} variables! Note that the substitution rules do not change the general form of the syzygy (e.g., all equations are the difference of two monomials involving two variables), and so the above theorem implies that there are only finitely many "forms" of syzygies.

1.2. **The first theorem.** We now give a more formal version of the above discussion. We begin by explaining precisely what the "master equation" is in coordinate-free language. Once this is accomplished, generalizing to higher syzygies will be straightforward. Let $I_n(V_1, \ldots, V_n)$ be the ideal defining the Segre embedding of $\mathbf{P}(V_1) \times \cdots \times \mathbf{P}(V_n)$. The n subscript simply indicates that we are dealing with n vector spaces; it is redundant but provides clarity. The ideal is generated by its degree two piece, so we focus on it; we denote it by $I_n^{(2)}(V_1, \ldots, V_n)$. We now observe that $I_n^{(2)}$ has the following pieces of structure:

(A1) Given maps $V_i \to V_i'$, there is a natural map $I_n^{(2)}(V_1, \ldots, V_n) \to I_n^{(2)}(V_1', \ldots, V_n')$. In other words, $I_n^{(2)}$ is a functor.

(A2) Given $\sigma \in S_n$, there is a natural map $\sigma_* : I_n^{(2)}(V_1, \ldots, V_n) \to I_n^{(2)}(V_{\sigma(1)}, \ldots, V_{\sigma(n)})$. In other words, the functor $I_n^{(2)}$ is S_n-equivariant.

(A3) There is a natural map

$$I_n^{(2)}(V_1, \ldots, V_{n-1}, V_n \otimes V_{n+1}) \to I_{n+1}^{(2)}(V_1, \ldots, V_{n-1}, V_n, V_{n+1}).$$

Geometrically, this map comes from the commutative diagram of Segre embeddings:

$$\mathbf{P}(V_1) \times \cdots \times \mathbf{P}(V_{n-1}) \times \mathbf{P}(V_n) \times \mathbf{P}(V_{n+1}) \longrightarrow \mathbf{P}(V_1) \times \cdots \times \mathbf{P}(V_{n-1}) \times \mathbf{P}(V_n \otimes V_{n+1})$$

$$\mathbf{P}(V_1 \otimes \cdots \otimes V_{n-1} \otimes V_n \otimes V_{n-1})$$

Note that (A3) seems to favor the nth parameter of I_n; however, due to (A2) symmetry is not actually broken.

The Segre embedding of $\mathbf{P}^1 \times \mathbf{P}^1$ into $\mathbf{P}^3$ is defined by a single quadratic equation, and so the space $I_2^{(2)}(\mathbf{C}^2, \mathbf{C}^2)$ is one dimensional. The "master equation" is simply a non-zero element of this space. As we now explain, any quadratic equation for any Segre embedding can be obtained from the master equation through the operations (A1)–(A3). Let $\{x_1, x_3\}$ be a basis for the first $\mathbf{C}^2$ and $\{x_2, x_4\}$ a basis for the second $\mathbf{C}^2$, so that $\{x_{12}, x_{14}, x_{32}, x_{34}\}$ is a basis for $\mathbf{C}^2 \otimes \mathbf{C}^2$. Then the quadratic polynomial

$$(1) \qquad x_{12}x_{34} - x_{14}x_{32}$$

is a non-zero element of $I_2^{(2)}(\mathbf{C}^2, \mathbf{C}^2)$, and can be taken as the master equation. Let $V_1, \ldots, V_n$ be vector spaces and let $\{x_j^{(k)}\}_{j \in I_k}$ be a basis for V_k. Let $A \amalg B$ be a partition of $\{1, \ldots, n\}$, let α and γ be elements of $\prod_{k \in A} I_k$ and let β and δ be elements of $\prod_{k \in B} I_k$. Then, in the obvious notation,

$$(2) \qquad x_{\alpha\beta}x_{\gamma\delta} - x_{\alpha\delta}x_{\gamma\beta}$$

is an element of $I_n^{(2)}(V_1, \ldots, V_n)$, and such elements span $I_n^{(2)}(V_1, \ldots, V_n)$. Consider the composite map

$$(3) \qquad I_2^{(2)}(\mathbf{C}^2, \mathbf{C}^2) \to I_2^{(2)}\left(\bigotimes_{k \in A} V_k, \bigotimes_{k \in B} V_k \right) \to I_n^{(2)}(V_1, \ldots, V_n).$$

The first map in (3) is the one coming form (A1) applied to the maps

$$f : \mathbf{C}^2 \to \bigotimes_{k \in A} V_k, \qquad f(x_1) = x_\alpha, \quad f(x_3) = x_\gamma$$

$$g : \mathbf{C}^2 \to \bigotimes_{k \in B} V_k, \qquad g(x_2) = x_\beta, \quad g(x_4) = x_\delta.$$

The second map in (3) comes from repeated applications of (A2) and (A3). The composite map in (3) simply takes an equation in the variables x_{12}, x_{14}, x_{32}, x_{34} and changes the 1 and 3 to α and γ and the 2 and 4 to β and δ. Thus the operations (A1)–(A3) truly do provide a coordinate-free replacement for the substitution rules of §1.1. In particular, the master equation (1) is taken to the equation (2). This shows that any quadratic equation can be obtained from the master equation through the operations (A1)–(A3).

As stated, generalizing the above discussion on equations to higher syzygies is straightforward. However, it will be convenient to first introduce a definition. A Δ-*module* is an object F consisting of:

(B1) For each non-negative integer n, an S_n-equivariant functor $F_n : \text{Vec}^n \to \text{Vec}$ satisfying a mild technical condition. Here, Vec denotes the category of complex vector spaces.

(B2) For each n, a natural transformation $F_n(V_1, \ldots, V_{n-1}, V_n \otimes V_{n+1}) \to F_{n+1}(V_1, \ldots, V_{n+1})$.

There are certain conditions placed on the maps in (B2) which we do not state here. It is clear how the above definition relates to the previous discussion: $I^{(2)}$ is a Δ-module. (Of course, (B1) takes the place of both (A1) and (A2), while (B2) corresponds to (A3).) However, this definition is inelegant, and thus technically inconvenient. An equivalent (but much nicer) definition is the following. Let Vec^Δ be the following category:

- An object is a pair (V, L) consisting of a finite set L and, for each $x \in L$, a vector space V_x. We think of (V, L) as a family of vector spaces.
- A morphism $(V, L) \to (V', L')$ consists of a surjection $L' \to L$ together with a linear map $V_x \to \bigotimes_{y \mapsto x} V'_y$, for each $x \in L$.

A Δ-module is then simply a functor $F : \text{Vec}^\Delta \to \text{Vec}$ (satisfying a mild technical condition). The functor F_n in the previous formulation is obtained by restricting F to families with n elements.

We now precisely define our spaces of syzygies and explain how they admit the structure of a Δ-module. Let (V, L) be a family of vector spaces. Let $R(V, L)$ be the projective coordinate ring of $\prod_{x \in L} \mathbf{P}(V_x)$ and let $P(V, L)$ be the projective coordinate ring of $\mathbf{P}(\bigotimes_{x \in L} V_x)$. Precisely,

$$P(V, L) = \bigoplus_{n=0}^{\infty} P^{(n)}(V, L), \qquad P^{(n)}(V, L) = \text{Sym}^n \left(\bigotimes_{x \in L} V_x \right)$$

$$R(V, L) = \bigoplus_{n=0}^{\infty} R^{(n)}(V, L), \qquad R^{(n)}(V, L) = \bigotimes_{x \in L} \text{Sym}^n(V_x).$$

The Segre embedding corresponds to a surjection of rings $P(V, L) \to R(V, L)$. Put

$$F_p(V, L) = \text{Tor}_p^{P(V,L)}(R(V, L), \mathbf{C}),$$

where $\mathbf{C}$ is given the structure of a $P(V, L)$-module by letting all positive degree elements act by 0. We call $F_p(V, L)$ the space of p-syzygies. This terminology is justified for the following reason: if

$$\cdots M_1 \to M_0 \to R(V, L) \to 0$$

is a minimal free resolution of $R(V, L)$ by finite free $P(V, L)$-modules, then M_p is isomorphic to $P(V, L) \otimes_{\mathbf{C}} F_p(V, L)$. (This is a general fact about minimal free resolutions and has nothing to do with the specifics of the Segre embedding.) The functoriality of F_p with respect to maps in Vec^Δ stems from a corresponding functoriality of R and P (as mentioned in the discussion of the

properties (A1)–(A3)), and the basic properties of Tor. Thus each F_p is a Δ-module. Note that F_1 is identified with $I^{(2)}$.

Before precisely stating our first theorem, we must make one more definition. Let F be a Δ-module. An *element* of F is an element of $F(V, L)$ for some (V, L). Given a set S of elements of F, there is a minimal Δ-submodule F' of F containing S. We call F' the submodule of F *generated* by S. We say that F is *finitely generated* if it is generated by some finite set of elements. We now come to our first main theorem:

Theorem A. *The Δ-module F_p is finitely generated.*

Of course, the "master syzygies" are just generators for F_p. As previously mentioned, our proof of this theorem provides an algorithm to compute a set of generators.

1.3. **The second theorem.** Theorem A tells us that all p-syzygies can be derived from a finite list of p-syzygies via some simple operations. However, this by itself tells us very little about the structure of the space $F_p(V, L)$ for particular values of (V, L). Our second theorem attempts to close this gap. The full version of the theorem requires some set-up, so we begin by stating a simple, elementary version of it:

Theorem B (Preliminary version)**.** *Fix non-negative integers p and d. Then the series*

$$\sum_{n=1}^{\infty} \dim F_{p,n}(\mathbf{C}^d, \ldots, \mathbf{C}^d) \cdot t^n$$

is a rational function.

The series in the above theorem counts the number of p-syzygies of the Segre embedding of $(\mathbf{P}^{d-1})^n$. Thus the result states that these numbers vary in a predictable manner with n.

The above preliminary version of the theorem falls short of what we would like, in three ways. First, it only deals with syzygies of $(\mathbf{P}^{d-1})^n$, rather than more general products. Second, it only gives us the dimension of $F_{p,n}$, rather than the dimensions of its graded pieces. And third, and most importantly, it gives no information on the action of the general linear groups on the spaces of syzygies. The full version of the theorem addresses all of these issues.

Before properly stating the theorem, we must introduce a generating function. For a partition λ, write $\mathbf{S}_\lambda$ for the corresponding Schur functor. (See §2.1 for a review of this theory.) By general theory, we have a decomposition

$$F_{p,n}(V_1, \ldots, V_n) = \bigoplus_{i \in I_{p,n}} \mathbf{S}_{\lambda_{1,i}}(V_1) \otimes \cdots \otimes \mathbf{S}_{\lambda_{n,i}}(V_n)$$

for some index set $I_{p,n}$ and partitions $\lambda_{i,j}$ (depending on n and p). It is not difficult to show that $I_{p,n}$ is finite. The above decomposition holds as functors $\mathrm{Vec}^n \to \mathrm{Vec}$, i.e, the set $I_{p,n}$ and the partitions $\lambda_{i,j}$ do not depend on $(V_1, \ldots, V_n)$. Define

$$f_{p,n}^* = \sum_{i \in I_{p,n}} s_{\lambda_{1,i}} \cdots s_{\lambda_{n,i}},$$

regarded as a polynomial in (commuting) formal variables s_λ. The functor $F_{p,n}$ can be recovered from $f_{p,n}^*$ — allowing the variables to commute does not lose information since the functor $F_{p,n}$ has an S_n-equivariance. (Note, however, that the data of the S_n-equivariance is not recorded in $f_{p,n}^*$.) Now define

$$f_p^* = \sum_{n=1}^{\infty} f_{p,n}^*,$$

a power series in the variables s_λ. A priori, there could be infinitely many variables occurring in f_p^*; we will show that this is not the case. Our second theorem is then:

Theorem B. *The series f_p^* is a rational function of the s_λ.*

This theorem completely encompasses the preliminary version, but is much stronger. Let us give an example to illustrate how to extract information from f_p^*. From the computations of §5.5, we obtain

$$f_1^* = \frac{1-s}{(1-s)^2 - w^2} - \frac{1}{1-s},$$

where $s = s_{(2)}$ and $w = s_{(1,1)}$. Developing the above into a power series, we obtain

$$f_1^* = w^2 + 3sw^2 + (6w^2 s^2 + w^4) + \cdots$$

The second order term of this series tells us that $F_{1,2}(V_1, V_2)$ is isomorphic to $\bigwedge^2 V_1 \otimes \bigwedge^2 V_2$. The third order terms tells us that $F_{1,3}(V_1, V_2, V_3)$ is isomorphic to

$$\left(\mathrm{Sym}^2 V_1 \otimes \bigwedge^2 V_2 \otimes \bigwedge^2 V_3 \right) \oplus \left(\bigwedge^2 V_1 \otimes \mathrm{Sym}^2 V_2 \otimes \bigwedge^2 V_3 \right) \oplus \left(\bigwedge^2 V_1 \otimes \bigwedge^2 V_2 \otimes \mathrm{Sym}^2 V_3 \right).$$

In general, we can recover the isomorphism class of $F_{p,n}(V_1, \ldots, V_n)$ as a representation of $\mathrm{GL}(V_1) \times \cdots \times \mathrm{GL}(V_n)$ from the order n term of $f_{p,n}^*$. In particular, we can recover the dimensions of the graded pieces of $F_{p,n}(V_1, \ldots, V_n)$ from $f_{p,n}^*$, as these pieces can be detected by the group action.

As with Theorem A, the proof of Theorem B provides an algorithm to compute the numerator and denominator of f_p^*. Thus, essentially all information about all p-syzygies of Segre embeddings can be determined from a single straightforward calculation. Unfortunately, this algorithm has complexity comparable to the previous one and is worthless in practice.

The leading term of f_p^* is known by Lascoux's work (§5.2). We will compute a certain Euler characteristic involving the f_p^*'s (§5.4). By known vanishing results, this allows us to compute f_1^*, f_2^*, f_3^* and part of f_4^* (§5.5). In particular, this gives an essentially complete description of the module of p-syzygies for $p = 1, 2, 3$. We have not been able to compute beyond these examples, however.

1.4. **Syzygies of Δ-schemes.** Given a finite family of vector spaces $(V_1, \ldots, V_n)$ we have the Segre variety $X_n(V_1, \ldots, V_n) \subset \mathbf{P}(V_1 \otimes \cdots \otimes V_n)$. The variety X_n is functorial in the V_i's and S_n-equivariant. Furthermore, we have inclusions

$$(4) \qquad\qquad X_{n+1}(V_1, \ldots, V_{n+1}) \subset X_n(V_1, \ldots, V_{n-1}, V_n \otimes V_{n+1}).$$

These are in fact the only properties of the Segre embedding we need for our theory to apply, as we now explain.

A Δ-*scheme* X is a rule assigning to each finite family of vector spaces $(V_1, \ldots, V_n)$ a closed subscheme $X_n(V_1, \ldots, V_n)$ of $\mathbf{P}(V_1 \otimes \cdots \otimes V_n)$ which is functorial, S_n-equivariant and such that the inclusion (4) holds. The Segre embeddings constitute an example of a Δ-scheme, but there are many others: for instance, the secant varieties of the Segre are Δ-schemes.

Let X be a Δ-scheme. Let $F_p(V, L)$ be the space of p-syzygies of the embedding $X(V, L) \to \mathbf{P}(\bigotimes_{x \in L} V_x)$. Then F_p forms a Δ-module. Let $F_p^{(d)}$ denote the degree d piece of F_p and let $(f_p^*)^{(d)}$ denote the series defined from $F_p^{(d)}$. Then we have the following result:

Theorem C. *The Δ-module $F_p^{(d)}$ is finitely generated, and the series $(f_p^*)^{(d)}$ is rational.*

This result is slightly weaker than the two main theorems for Segre embeddings, since it only applies to the graded pieces of F_p. To get comparable results, one would need to know that F_p is supported in only finitely many degrees. This is true in the case of Segres, but probably not true for a general Δ-scheme. In §4.5 we define a class of Δ-schemes, those of *finite level*, for which it seems plausible that F_p is supported in finitely many degrees.

1.5. **Outline of proofs.** We begin with a general study of Δ-modules. The heart of the paper is occupied with the proof of two theorems: (a) a small Δ-module is noetherian; and (b) the Hilbert series of a small Δ-module is rational. "Small" Δ-modules form a certain subclass of Δ-modules. The class of small Δ-modules is closed under subquotients. Theorem C is then a formal consequence of (a) and (b) and the following observation: there is a complex of small Δ-modules (a Koszul complex) whose homology is F_p. The two stronger theorems for Segre embeddings are obtained by applying the well-known result that the p-syzygies of the Segre embedding are supported in only finitely many degrees.

We have not been able to find a direct method of studying Δ-modules. Instead, we access them through the following "ladder":

$$\{\text{graded modules}\} \leftrightarrow \{\text{modules in Sym(Vec)}\} \leftrightarrow \{\text{modules in Sym}(\mathcal{S})\} \leftrightarrow \{\Delta\text{-modules}\}$$

We define the middle two categories below. The arrows here do not mean anything precise, only that the two categories are related. We prove our results about Δ-modules by starting with the corresponding results for graded modules (which are easy and well-known) and moving up the ladder. This process is actually fairly explicit: for example, the Hilbert series of a Δ-module is identified with the (G-equivariant) Hilbert series of a module over a graded ring, which is constructed in an explicit manner from the Δ-module. This identification is the reason that the Hilbert series of a Δ-module — at least one which is described in a reasonable manner — is algorithmically computable.

1.6. **Future directions.** We prove a number of purely algebraic results (about Δ-module, twisted commutative algebras, etc.) which seem to scratch the surface of a much larger theory. We hope to pursue some of this theory in the future. We also hope to refine our results on syzygies — for instance, what can one say about the rational functions f_p^*? A list of some questions and problems along these lines appears in §6.

Some of the results and methods we develop seem to apply to slightly different settings. For example, if X is a projective variety with an action of a reductive group G, then the syzygies of the GIT quotients $X^n/\!/G$ (as n varies) have some of the algebraic structure that we observe here for the Segre varieties. One can hope that there are finiteness results in this setting. We will return to this topic in a future paper.

2. Δ-modules

The purpose of this section is to introduce the algebraic objects that we will use in the rest of the paper, most notably Δ-modules. We begin by quickly reviewing the theory of Schur functors and define the Schur algebra $\mathcal{S}$, which we regard as a category. We then discuss symmetric powers of semi-simple abelian categories. This operation is not strictly necessary for our purposes, but clarifies some later definitions. We then introduce the categories Sym(Vec) and Sym($\mathcal{S}$) and give explicit models for them which do not use the symmetric power construction. Finally we discuss Δ-modules, which are objects of Sym($\mathcal{S}$) with an additional piece of structure. The most important results of this section are various statements that certain types of objects are noetherian.

Algebras in Sym(Vec) are known as "twisted commutative algebras," and there is some literature on them (see [B], [J, Ch. 4], [GS]); closely related is Joyal's theory of tensorial species. However, the existing literature — at least that which we are aware of — studies these objects from a perspective different from our own. In particular, we have not encountered the noetherian result

we prove about them in the literature. The categories $\mathrm{Sym}(\mathcal{S})$ and Mod_Δ are a bit more esoteric, and we do not know of any occurrence of them in the literature.

2.1. **The Schur algebra.** We quickly review what we need about the Schur algebra. Further discussion can be found in [FH, §6.1]. Let Vec denote the category of complex vector spaces. A functor $\mathrm{Vec} \to \mathrm{Vec}$ is "nice" if it appears as a constituent of a functor of the form $V \mapsto V^{\otimes n}$, or a (possibly infinite) direct sum of such functors. Essentially every functor $\mathrm{Vec} \to \mathrm{Vec}$ one encounters naturally that does not use duality is nice; an example of a non-nice functor is the double dual. Let $\mathcal{S}$ denote the category of all nice functors. It is a semi-simple abelian tensor category which we call the *Schur algebra*. (By "semi-simple" here we mean that every object is a possibly infinite direct sum of simple objects.) The tensor product is the point-wise one: $(F \otimes G)(V) = F(V) \otimes G(V)$.

Let λ be a partition of n; we denote this by $|\lambda| = n$ or $\lambda \vdash n$. Let $\mathbf{M}_\lambda$ be the irreducible complex representation of the symmetric group S_n associated to λ. For a vector space V put $\mathbf{S}_\lambda(V) = (V^{\otimes n} \otimes \mathbf{M}_\lambda)_{S_n}$, where the subscript denotes coinvariants and S_n acts on $V^{\otimes n}$ by permuting tensor factors. Then $\mathbf{S}_\lambda$ belongs to $\mathcal{S}$ and is a simple object; furthermore, every simple object is isomorphic to $\mathbf{S}_\lambda$ for a unique λ. We use the convention that the partition $\lambda = (n)$ gives the functor $\mathbf{S}_\lambda = \mathrm{Sym}^n$ while the partition $\lambda = (1, \ldots, 1)$ gives the functor $\mathbf{S}_\lambda = \bigwedge^n$. We identify partitions with Young diagrams by the convention that $\lambda = (n)$ has one row and $\lambda = (1, \ldots, 1)$ has n rows.

We will also need multivariate functors of vector spaces. A functor $\mathrm{Vec}^k \to \mathrm{Vec}$ is "nice" if it appears as a constituent of a functor of the form $(V_1, \ldots, V_k) \mapsto V_1^{\otimes n_1} \otimes \cdots \otimes V_k^{\otimes n_k}$, or a (possibly infinite) direct sum of such functors. The category of all nice functors is again a semi-simple abelian tensor category, and is naturally equivalent to $\mathcal{S}^{\otimes k}$. This means that any nice functor $F : \mathrm{Vec}^k \to \mathrm{Vec}$ can be written as

$$F(V_1, \ldots, V_k) = \bigoplus_{i \in I} \mathbf{S}_{\lambda_{1,i}}(V_1) \otimes \cdots \otimes \mathbf{S}_{\lambda_{k,i}}(V_k)$$

for some index set I and partitions $\lambda_{i,j}$. This expression is unique. We say that a partition λ *occurs* in F if it is amongst the $\lambda_{i,j}$. If $F : \mathrm{Vec} \to \mathrm{Vec}$ is a nice functor then the functors $\mathrm{Vec}^2 \to \mathrm{Vec}$ given by mapping (V, W) to $F(V \oplus W)$ and $F(V \otimes W)$ are both nice. We thus get a co-addition map a^* and a co-multiplication map m^* from $\mathcal{S}$ to $\mathcal{S}^{\otimes 2}$.

Let $\mathcal{A}$ be a $\mathbf{C}$-linear abelian tensor category. (Our tensor categories are always symmetric, i.e., there is a given involution of functors $A \otimes B \to B \otimes A$.) There is then a natural action of $\mathcal{S}$ on $\mathcal{A}$. Given a partition λ of n and an object A of $\mathcal{A}$, the object $\mathbf{S}_\lambda(A)$ is given by $(A^{\otimes n} \otimes \mathbf{M}_\lambda)_{S_n}$.

2.2. **Symmetric powers of abelian categories.** We now describe how one can form the symmetric algebra on semi-simple abelian categories. The reader who is not comfortable with this discussion need not worry: all categories that we eventually use will admit concrete descriptions. We provide this discussion only to give some context and motivation for later constructions.

Let $\mathcal{A}$ be a semi-simple $\mathbf{C}$-linear abelian category. One can then make sense of the tensor power $\mathcal{A}^{\otimes n}$ (see [D, §5] for a discussion of tensor products of abelian categories). We define $\mathrm{Sym}^n(\mathcal{A})$ to be the category $(\mathcal{A}^{\otimes n})^{S_n}$, where the superscript denotes homotopy invariants (equivariant objects). Thus an object of $\mathrm{Sym}^n(\mathcal{A})$ is an object A of $\mathcal{A}^{\otimes n}$ together with an isomorphism $\sigma^* A \to A$ for each $\sigma \in S_n$, satisfying the obvious compatibility conditions. We define $\mathrm{Sym}(\mathcal{A})$ to be the sum of the categories $\mathrm{Sym}^n(\mathcal{A})$ over $n \geq 0$.

The category $\mathrm{Sym}(\mathcal{A})$ has a natural tensor structure. Multiplication involves averaging (induction). Precisely, let $\underline{\otimes}$ denote the usual concatenation tensor product $\mathcal{A}^{\otimes n} \otimes \mathcal{A}^{\otimes m} \to \mathcal{A}^{\otimes(n+m)}$. If A is an object of $(\mathcal{A}^{\overline{\otimes n}})^{S_n}$ and B of $(\mathcal{A}^{\otimes m})^{S_m}$ then $A \underline{\otimes} B$ naturally has an $S_n \times S_m$ equivariance. One can then form the induction $\mathrm{Ind}_{S_n \times S_m}^{S_{n+m}}(A \underline{\otimes} B)$, which is an object of $(\mathcal{A}^{\otimes(n+m)})^{S_{n+m}}$. This is the product of A and B in $\mathrm{Sym}(\mathcal{A})$, which we denote by $A \otimes B$. Note that if $\mathcal{A}$ itself has a tensor structure then $(\mathcal{A}^{\otimes n})^{S_n}$ does as well, which gives an alternate tensor structure on $\mathrm{Sym}(\mathcal{A})$. When present, we will denote this tensor product by $\boxtimes$ and call it the *point-wise* tensor product.

Remark 2.1. The above definition looks more like the divided power algebra than symmetric algebra. However, one can verify it has the correct universal property. We believe that in the setting of $\mathbf{C}$-linear abelian tensor categories, the right analogue of divided powers is an action of the Schur algebra, with Sym^n taking the place of γ_n. Since all such categories have a natural action of the Schur algebra, they can all be considered to have divided powers. Thus the symmetric and divided power algebras on $\mathcal{A}$ coincide. This is a conceptual reason explaining why we can use invariants (rather than coinvariants) to form the symmetric algebra.

As $\mathrm{Sym}(\mathcal{A})$ is an abelian tensor category, we can speak of algebras in it and modules over algebras. (Algebras will always be commutative, associative and unital.) Let A be an algebra. We say that A is *finitely generated* (as an algebra) if it is a quotient of $\mathrm{Sym}(F)$ for some finite length object F of $\mathrm{Sym}(\mathcal{A})$. Similarly, we say that an A-module M is *finitely generated* if it is a quotient of $A \otimes F$ for some finite length F. We say that M is *noetherian* if every ascending chain of submodules stabilizes. We say that A is *noetherian* (as an algebra) if every finitely generated A-module is noetherian.

The category $\mathrm{Sym}(\mathcal{A})$ is again semi-simple abelian, and its simple objects can be described easily. Let A be a simple object of $\mathcal{A}$ and let $\mathbf{M}_\lambda$ be an irreducible representation of S_n. Then $\mathbf{M}_\lambda \otimes A^{\underline{\otimes} n}$ has a natural S_n-equivariance and so defines an object of $\mathrm{Sym}^n(\mathcal{A})$; in fact, this is nothing other than $\mathbf{S}_\lambda(A)$ in the category $\mathrm{Sym}(\mathcal{A})$. This object is simple, and all simple objects are of the form $\bigotimes \mathbf{S}_{\lambda_i}(A_i)$ where the A_i are mutually non-isomorphic simple objects of $\mathcal{A}$.

Let $K(-)$ denote the Grothendieck group of an abelian category, tensored with $\mathbf{Q}$. We have a map

$$K(\mathrm{Sym}^n(\mathcal{A})) \to \mathrm{Sym}^n(K(\mathcal{A}))$$

defined as follows: first apply the functor $\mathrm{Sym}^n(\mathcal{A}) \to \mathcal{A}^{\otimes n}$, then use the identification $K(\mathcal{A}^{\otimes n}) = K(\mathcal{A})^{\otimes n}$, then project $K(\mathcal{A})^{\otimes n} \to \mathrm{Sym}^n(K(\mathcal{A}))$ and finally divide by $n!$. For $A \in \mathrm{Sym}^n(\mathcal{A})$ we let $[A]$ denote the corresponding class in $\mathrm{Sym}^n(K(\mathcal{A}))$. We also put $[A]^* = n![A]$. The above map is not injective, but only forgets the S_n-equivariant structure: if $[A] = [B]$ then A and B have isomorphic images in $\mathcal{A}^{\otimes n}$. Summing the above maps over n, we get a map

$$K(\mathrm{Sym}(\mathcal{A})) \to \mathrm{Sym}(K(\mathcal{A})).$$

Again, for $A \in \mathrm{Sym}(\mathcal{A})$ we let $[A]$ denote the corresponding class in $\mathrm{Sym}(K(\mathcal{A}))$; we define $[A]^*$ in the obvious manner. We have $[A \otimes B] = [A][B]$; the modified class $[-]^*$ does not satisfy this. As an example, if A is an object of $\mathcal{A}$ then $[\mathrm{Sym}^n(A)] = \frac{1}{n!}[A]^n$ (which leads to the beautiful formula $[\mathrm{Sym}(A)] = \exp([A])$).

We will often need to use the completion of Sym, both in the setting of $\mathbf{Z}$-modules and abelian categories. Whereas $\mathrm{Sym}(\mathbf{Z}^n)$ is the ring of polynomials in n variables, the completion is the ring of power series in n variables. We will not bother to introduce extra notation for the completion, as there should be no confusion.

2.3. **The category** $\mathrm{Sym}(\mathrm{Vec})$. The following abelian tensor categories are equivalent:

(a) The category $\mathrm{Sym}(\mathrm{Vec})$.
(b) The category of functors (fs) $\to$ Vec, where (fs) is the category whose objects are finite sets and whose morphisms are bijections. Multiplication is given by convolution, using the monoidal structure on (fs) given by disjoint union. Precisely, if V and W are two functors (fs) $\to$ Vec then

$$(V \otimes W)_L = \bigoplus_{L=A \amalg B} V_A \otimes W_B,$$

where the sum is over all partitions of L into two disjoint subsets A and B. (We use subscripts to indicate the value of the functor on a set).

(c) The category of sequences $(V_n)_{n\geq 0}$, where V_n is a vector space with an action of S_n. Multiplication is given by the formula

$$(V \otimes W)_n = \bigoplus_{n=i+j} \operatorname{Ind}_{S_i \times S_j}^{S_n} (V_i \otimes W_j).$$

(d) The Schur algebra $\mathcal{S}$. Multiplication is the point-wise tensor product.

(e) The full subcategory of the representation category of $\operatorname{GL}(\infty)$ on objects which appear as a constituent of a direct sum of tensor powers of the standard representation $\mathbf{C}^\infty$. Here $\operatorname{GL}(\infty)$ is the union of $\operatorname{GL}(n, \mathbf{C})$ over $n \geq 1$. Multiplication is the usual tensor product of representations.

We briefly describe the various equivalences. The equivalence between (b) and (c) is clear. Since $\operatorname{Vec}^{\otimes n} = \operatorname{Vec}$, the category of S_n-equivariant objects in $\operatorname{Vec}^{\otimes n}$ is just the representation category of S_n; this gives the equivalence between (a) and (c). The equivalence between (c) and (d) is through Schur-Weyl duality. Precisely, given a sequence (V_n) in the category (c), let $S : \operatorname{Vec} \to \operatorname{Vec}$ be the functor taking a vector space W to

$$S(W) = \bigoplus_{n \geq 0} (W^{\otimes n} \otimes V_n)_{S_n}.$$

Then $(V_n) \mapsto S$ is the equivalence. Regarding $\mathbf{M}_\lambda$ as an object of (c) supported at $n = |\lambda|$, this equivalence maps $\mathbf{M}_\lambda$ to $\mathbf{S}_\lambda$. Finally, the equivalence of (d) and (e) is given by evaluating a Schur functor on $\mathbf{C}^\infty$. We regard (b)–(e) as "models" for the category $\operatorname{Sym}(\operatorname{Vec})$. We name them the "standard," "sequence," "Schur," and "GL" models, respectively. (There are even more models for this category, but they are not relevant for our purposes.) We now come to an important definition:

Definition 2.2. A *twisted commutative algebra* is an algebra in the category $\operatorname{Sym}(\operatorname{Vec})$.

As always, "algebra" means commutative, associative and unital. In the standard model, a twisted commutative algebra is a functor $A : (\mathrm{fs}) \to \operatorname{Vec}$ together with a multiplication map $A_L \otimes A_{L'} \to A_{L \amalg L'}$ satisfying the appropriate identity. In the GL-model, a twisted commutative algebra is just a commutative $\mathbf{C}$-algebra, in the usual sense, with an action of $\operatorname{GL}(\infty)$ by algebra homomorphisms (under which the algebra forms an appropriate kind of representation). The notion of a module over a twisted commutative algebra is evident.

Let V be an object of $\operatorname{Sym}(\operatorname{Vec})$, in the standard model. By an *element* of V we mean an element of V_L for some L; we then call $\#L$ its *order*. (It would be more natural to use the term "degree" here, but we want to reserve that term for future use.) Given a collection S of elements of V there is a unique minimal subobject of V containing S; we call it the subspace of V *generated* by S. Similarly, if A is a twisted commutative algebra and S is a collection of elements of A then one can speak of the subalgebra of A generated by S. One can do the same for modules over A. We therefore have a notion of "finitely generated" for algebras and modules; this agrees with the one defined in §2.2 using finite length objects.

Let U be a vector space. We let $U\langle 1 \rangle$ be the object of $\operatorname{Sym}(\operatorname{Vec})$ which is U in order 1 and 0 in other orders. We put $A = \operatorname{Sym}(U\langle 1 \rangle)$; this is the most important twisted commutative algebra in this paper. Here are precise descriptions of $U\langle 1 \rangle$ and A in the various models:

- In the standard model, $U\langle 1 \rangle$ assigns to a finite set L the vector space U if $\#L = 1$ and the vector space 0 otherwise. The algebra A assigns to a finite set L the vector space $U^{\otimes L}$. The multiplication map $A_L \otimes A_{L'} \to A_{L \amalg L'}$ is concatenation of tensors. (Note: $U^{\otimes L}$ is isomorphic as a vector space to $U^{\otimes n}$, where $n = \#L$, but is functorial in L. It can be defined as the universal vector space equipped with a multi-linear map from $U \times L$. We think of the factors of a pure tensor in $U^{\otimes L}$ as being labeled with elements of L.)

- In the sequence model, $U\langle 1 \rangle$ is the vector space U in order 1 and 0 in other orders. The algebra A is given by $A_n = U^{\otimes n}$, with its usual S_n action; the multiplication map $A_i \otimes A_j \to A_{i+j}$ is again concatenation of tensors.

- In the Schur model, $U\langle 1\rangle$ is the functor $U\otimes\mathrm{Sym}^1$. The algebra A is given by $\mathrm{Sym}(U\otimes\mathrm{Sym}^1)$. If V is a vector space then $(U\langle 1\rangle)(V) = U\otimes V$ and $A(V) = \mathrm{Sym}(U\otimes V)$.
- In the GL-model, $U\langle 1\rangle$ is $U\otimes\mathbf{C}^\infty$. The algebra A is $\mathrm{Sym}(U\otimes\mathbf{C}^\infty)$. Thus, in this model, A is a polynomial ring in infinitely many variables with $\mathrm{GL}(\infty)$ acting by linear substitutions.

The following result underlies everything else in the paper. We learned the key ideas of its proof from Harm Derksen and Ben Howard.

Theorem 2.3. *A twisted commutative algebra finitely generated in order 1 is noetherian.*

Proof. A twisted commutative algebra which is finitely generated in order 1 is a quotient of $\mathrm{Sym}(U\langle 1\rangle)$ for some finite dimensional vector space U. Thus it suffices to show that such algebras are noetherian. Fix U and put $A = \mathrm{Sym}(U\langle 1\rangle)$. In the GL-model, A is given by $\mathrm{Sym}(U\otimes\mathbf{C}^\infty)$. Let $d = \dim U$. The following lemma shows that contraction from $\mathrm{GL}(\infty)$-stable ideals of $\mathrm{Sym}(U\otimes\mathbf{C}^\infty)$ to ideals of $\mathrm{Sym}(U\otimes\mathbf{C}^d)$ is injective. As $\mathrm{Sym}(U\otimes\mathbf{C}^d)$ is a polynomial algebra in finitely many variables, it is noetherian. We conclude that A is a noetherian module over itself. A slight modification of this argument shows that any finitely generated A-module is noetherian, which proves that A is noetherian as an algebra. $\square$

Lemma 2.4 (Weyl). *Let U be a vector space of dimension d. If $W \subset \mathrm{Sym}^k(U\otimes\mathbf{C}^\infty)$ is a $\mathrm{GL}(\infty)$-stable subspace then W is generated as a $\mathrm{GL}(\infty)$-module by $W\cap\mathrm{Sym}^k(U\otimes\mathbf{C}^d)$.*

Proof. Using the formula for the symmetric power of a tensor product (see [FH, Exercise 6.11(b)]), we obtain a diagram

$$
\begin{array}{ccc}
\mathrm{Sym}^k(U\otimes\mathbf{C}^d) & = & \bigoplus \mathbf{S}_\lambda(U)\otimes\mathbf{S}_\lambda(\mathbf{C}^d) \\
\uparrow & & \uparrow \\
\mathrm{Sym}^k(U\otimes\mathbf{C}^\infty) & = & \bigoplus \mathbf{S}_\lambda(U)\otimes\mathbf{S}_\lambda(\mathbf{C}^\infty)
\end{array}
$$

The sums are taken over the partitions λ of k. Now, let W be a $\mathrm{GL}(\infty)$-stable subspace of $\mathrm{Sym}^k(U\otimes\mathbf{C}^\infty)$. Since the $\mathbf{S}_\lambda(\mathbf{C}^\infty)$ are mutually non-isomorphic irreducible $\mathrm{GL}(\infty)$-modules, we can write $W = \bigoplus W_\lambda\otimes\mathbf{S}_\lambda(\mathbf{C}^\infty)$, where W_λ is a subspace of $\mathbf{S}_\lambda(U)$. Note that if λ has more than d rows then $\mathbf{S}_\lambda(U) = 0$ and so $W_\lambda = 0$. The space $W\cap\mathrm{Sym}^k(U\otimes\mathbf{C}^d)$ is equal to $\bigoplus W_\lambda\otimes\mathbf{S}_\lambda(\mathbf{C}^d)$. Since $\mathbf{S}_\lambda(\mathbf{C}^d)$ generates $\mathbf{S}_\lambda(\mathbf{C}^\infty)$ whenever λ has at most d rows, we find that $W\cap\mathrm{Sym}^k(U\otimes\mathbf{C}^d)$ generates W. $\square$

Any twisted commutative algebra finitely generated in order 1 has the property that the partitions appearing in it have a bounded number of rows. Most of the results we prove about algebras generated in order 1, such as the above theorem, can easily be extended to the larger class of algebras which have a bounded number of rows. Going beyond this class of algebras is harder though. For instance, we do not know how much the order 1 hypothesis in Theorem 2.3 can be relaxed; see §6. We need one more general finiteness result about twisted commutative algebras.

Proposition 2.5. *Let A be a noetherian twisted commutative algebra on which a reductive group G acts. Then A^G is noetherian. If M is a finitely generated A-module with a compatible action of G then M^G is a finitely generated A^G-module.*

Proof. The usual proof works. To show how it carries over, we prove that A^G is noetherian. We must therefore show that $A^G\otimes F$ is a noetherian A^G-module for any finite length object F of $\mathrm{Sym}(\mathrm{Vec})$. We have an inclusion $A^G\otimes F \subset A\otimes F$. Let G act on $A\otimes F$ by acting trivially on F. Let M be an A^G-submodule of $A^G\otimes F$ and let M' be the A-submodule of $A\otimes F$ it generates. Let y be an element of M' which is G-invariant. We can then write $y = \sum a_i x_i$ where the a_i are elements of A and the x_i are elements of M. As y and the x_i are invariant, we have $y = \mathrm{avg}(y) = \sum \mathrm{avg}(a_i)x_i$. Thus y belongs to M. This shows that $M = (M')^G$. Therefore, the map

$$\{A^G\text{-submodules of } A^G\otimes F\} \to \{A\text{-submodules of } A\otimes F\}$$

which takes an A^G-submodule to the A-submodule it generates is inclusion preserving and injective. As any ascending chain on the right side stabilizes, so too does any ascending chain on the left side. This proves that A^G is noetherian. We leave the statement about modules to the reader. $\square$

2.4. **The category** $\mathrm{Sym}(\mathcal{S})$. We now investigate the category $\mathrm{Sym}(\mathcal{S})$, which will feature prominently in what follows. We begin by giving a useful description of it. Let Vec^f denote the category of finite families of vector spaces. An object of Vec^f is a pair (V, L) where L is a finite set and V assigns to each element x of L a vector space V_x. A morphism $(V, L) \to (V', L')$ consists of a bijection $i : L \to L'$ and a linear map $V_x \to V'_{i(x)}$ for each $x \in L$. We now claim that the following three categories are equivalent:

  (a) The category $\mathrm{Sym}(\mathcal{S})$.
  (b) The category of nice functors $\mathrm{Vec}^f \to \mathrm{Vec}$.
  (c) The category of sequences (F_n) where F_n is a nice S_n-equivariant functor $\mathrm{Vec}^n \to \mathrm{Vec}$.

The equivalences, as well as the definition of "nice" in (b), should be clear. Each of these categories is an abelian tensor category and the various equivalences respect this structure. The tensor structure in (b) is given by

$$(F \otimes G)(V, L) = \bigoplus_{L = A \amalg B} F(V|_A, A) \otimes G(V|_B, B).$$

The tensor structure in (c) is like that for the sequence model of $\mathrm{Sym}(\mathrm{Vec})$. Note that since $\mathcal{S}$ is itself a tensor category we have a point-wise tensor product $\boxtimes$ on $\mathrm{Sym}(\mathcal{S})$. In (b) this product is given by

$$(F \boxtimes G)(V, L) = F(V, L) \otimes G(V, L),$$

hence the name "point-wise tensor product." This tensor product will not be used in the rest of this section but will show up later.

As $\mathrm{Sym}(\mathcal{S})$ is an abelian tensor category, we have the notions of algebras and modules over algebras in $\mathrm{Sym}(\mathcal{S})$. We note that for $A \in \mathrm{Sym}(\mathcal{S})$ giving a multiplication map $A \otimes A \to A$ amounts to giving a natural map

$$A(V, L) \otimes A(V', L') \to A(V \amalg V', L \amalg L'),$$

where $V \amalg V'$ denotes the natural map $L' \amalg L \to \mathrm{Vec}$ built out of V and V'. Define an *element* of $F \in \mathrm{Sym}(\mathcal{S})$ to be an element of $F(V, L)$ for some (V, L). We call $\#L$ the *order* of the element. (We will use the term "degree" later to reference the grading on $\mathcal{S}$.) As in the twisted commutative setting, one can then give an elemental definition for "finitely generated" and this agrees with the more general definition in terms of finite length objects given in §2.2.

We now give examples of some of the above definitions to give the reader some sense of their nature. Let F_λ be the object of $\mathrm{Sym}(\mathcal{S})$ which assigns to (V, L) the space 0 if $\#L \ne 1$ and the space $\mathbf{S}_\lambda(V_x)$ if $L = \{x\}$. Then $F_\lambda \otimes F_\mu$ assigns to (V, L) the space 0 if $\#L \ne 2$ and the space

$$[\mathbf{S}_\lambda(V_x) \otimes \mathbf{S}_\mu(V_y)] \oplus [\mathbf{S}_\lambda(V_y) \otimes \mathbf{S}_\mu(V_x)]$$

if $L = \{x, y\}$. As a second example, $\mathrm{Sym}(F_\lambda)$ is the object of $\mathrm{Sym}(\mathcal{S})$ which assigns to (V, L) the space $\bigotimes_{x \in L} \mathbf{S}_\lambda(V_x)$. Note in particular that the only partition appearing in the symmetric algebra $\mathrm{Sym}(F_\lambda)$ is λ itself. These examples underline the fact that the product in $\mathrm{Sym}(\mathcal{S})$ is formal: the Littlewood–Richardson rule does intervene in any way. (The Littlewood–Richardson rule is used in the point-wise product $\boxtimes$ for $\mathrm{Sym}(\mathcal{S})$.)

The following is the main result we need on algebras in $\mathrm{Sym}(\mathcal{S})$.

Theorem 2.6. *An algebra in $\mathrm{Sym}(\mathcal{S})$ which is finitely generated in order 1 is noetherian.*

Proof. Let A be an algebra finitely generated in order 1. Let F be a finite length object of $\mathrm{Sym}(\mathcal{S})$. We must show that $A \otimes F$ is a noetherian A-module. As with any finitely generated algebra in $\mathrm{Sym}(\mathcal{S})$, the number of rows in any partition appearing in A is bounded; the same holds for $A \otimes F$.

Let d be an integer such that any partition appearing in $A \otimes F$ has at most d rows, and let U be a vector space of dimension d. For a finite set L let U_L be the constant family on U, i.e., the object (V, L) of Vec^f with $V_x = U$ for all x. Then $L \mapsto U_L$ defines a functor $i : (\mathrm{fs}) \to \mathrm{Vec}^f$, which respects the monoidal structure (disjoint union) on each category. The induced functor $i^* : \mathrm{Sym}(\mathcal{S}) \to \mathrm{Sym}(\mathrm{Vec})$ is a tensor functor. One easily sees that if M' and M are two sub-objects of $A \otimes F$ then $M = M'$ if and only if $i^*M = i^*M'$. We have thus shown that the map

$$\{A\text{-submodules of } A \otimes F\} \to \{(i^*A)\text{-submodules of } i^*(A \otimes F)\}$$

is injective; it is obviously inclusion preserving. Now, i^*A is a twisted commutative algebra finitely generated in order 1; to see this, write A as a quotient of $\mathrm{Sym}(S)$ with $S \in \mathcal{S}$, so that i^*A is a quotient of $\mathrm{Sym}(S(U)\langle 1 \rangle)$. We thus find that i^*A is noetherian by Theorem 2.3. As $i^*(A \otimes F) = (i^*A) \otimes (i^*F)$ is a finitely generated (i^*A)-module, it is noetherian. We thus find that every ascending chain of A-submodules of $A \otimes F$ stabilizes, and so A is noetherian. $\square$

The following proposition is proved just like Proposition 2.5.

Proposition 2.7. *Let A be a noetherian algebra in $\mathrm{Sym}(\mathcal{S})$ on which a reductive group G acts. Then A^G is again noetherian. If M is a finitely generated A-module with a compatible action of G then M^G is a finitely generated A^G-module.*

2.5. Δ-modules. Let Vec^{Δ} be the category whose objects are pairs (V, L) as in Vec^f, but where now a morphism $(V, L) \to (V', L')$ consists of a surjection $L' \to L$ together with a map $V_x \to \bigotimes_{y \mapsto x} V'_y$ for each $x \in L$. We now come to a central concept in this paper:

Definition 2.8. A *Δ-module* is a nice functor $\mathrm{Vec}^{\Delta} \to \mathrm{Vec}$.

In the above definition, we say that a functor $\mathrm{Vec}^{\Delta} \to \mathrm{Vec}$ is nice if it is so when restricted to Vec^f. We denote the category of these functors by Mod_{Δ}. It is abelian, though not semi-simple. Since Vec^f is a sub-category of Vec^{Δ}, every Δ-module defines an object of $\mathrm{Sym}(\mathcal{S})$. For a Δ-module F we let $[F]$ be the class in $\mathrm{Sym}(K(\mathcal{S}))$ obtained by regarding F as an object of $\mathrm{Sym}(\mathcal{S})$.

The forgetful functor $\mathrm{Mod}_{\Delta} \to \mathrm{Sym}(\mathcal{S})$ has a left adjoint, which we denote by Φ. To describe this functor explicitly, we must introduce a piece of notation. Let (V, L) be an object of Vec^f and let $\mathscr{U}$ be a partition of L. For a set $S \in \mathscr{U}$ put $V_S = \bigotimes_{x \in S} V_x$. Then $(V, \mathscr{U})$ is an object of Vec^f. There is a natural map $(V, \mathscr{U}) \to (V, L)$ in Vec^{Δ}, the surjection $L \to \mathscr{U}$ mapping an element of L to the part of $\mathscr{U}$ to which it belongs. With this notation in hand, we can give the following formula for Φ:

$$(\Phi F)(V, L) = \bigoplus F(V, \mathscr{U}),$$

where the sum is over all partitions $\mathscr{U}$ of L. We leave it to the reader to work out the Δ-structure on $\Phi(F)$ and verify its universal property. We call $\Phi(F)$ the *free Δ-module* on F. By a *(finite) free Δ-module* we mean one isomorphic to $\Phi(F)$, where F is a (finite length) object in $\mathrm{Sym}(\mathcal{S})$. Free modules are projective since Φ is a left adjoint and $\mathrm{Sym}(\mathcal{S})$ is semi-simple.

We define an *element* of a Δ-module F to be an element of $F(V, L)$ for some (V, L). Given a collection of elements of F one can speak of the Δ-submodule that it generates. We say that F is *finitely generated* if there is a finite set of elements of F that generate it. This is equivalent to F being a quotient of a finite free Δ-module. We say that a Δ-module is *noetherian* if every ascending chain of Δ-submodules stabilizes. Note that noetherian implies finitely generated.

Let n be a positive integer. Let T_n be the object of $\mathrm{Sym}(\mathcal{S})$ which assigns to (V, L) the space $V_x^{\otimes n}$ if $L = \{x\}$ and 0 otherwise. Let W_n be the symmetric algebra on T_n in the category $\mathrm{Sym}(\mathcal{S})$. It is given by

$$W_n(V, L) = \bigotimes_{x \in L} V_x^{\otimes n}$$

The multiplication map in W_n is given by concatenation of tensors. Note that S_n acts on W_n by algebra homomorphisms. We also consider the free Δ-module on T_n. It is given by

$$(\Phi T_n)(V, L) = \bigotimes_{x \in L} V_x^{\otimes n}$$

if $\#L \geq 1$, while $(\Phi T_n)(V, L) = 0$ for $\#L = 0$. Observe that $\Phi(T_n)$ is naturally a subobject of W_n in the category $\mathrm{Sym}(\mathcal{S})$, and is in fact an ideal. The following is the key result connecting Δ-modules to objects we have previously studied, the final rung of the ladder of §1.5.

Proposition 2.9. *Any Δ-submodule of $\Phi(T_n)$ is a $W_n^{S_n}$-submodule of $\Phi(T_n)$.*

Proof. Let (V, L) and (V', L') be two objects of Vec^f. Let $W = \bigotimes_{y \in L'} V'_y$, so that $W_n(V', L') = W^{\otimes n}$. Let $v \in (\Phi T_n)(V, L)$, and let $v' \in W^{\otimes n}$ be S_n-invariant. We must show that the image of $v \otimes v'$ under the multiplication map $(\Phi T_n)(V, L) \otimes W^{\otimes n} \to (\Phi T_n)(V \amalg V', L \amalg L')$ belongs to the Δ-submodule of ΦT_n generated by v. Now, $(W^{\otimes n})^{S_n}$ is spanned by nth tensor powers of elements of W. It thus suffices to treat the case $v' = w^{\otimes n}$ for some $w \in W$.

Pick an element x_0 of L. Define a map $f : (V, L) \to (V \amalg V', L \amalg L')$ in Vec^Δ, as follows. The map $L \amalg L' \to L$ is the identity on L and collapses all of L' to x_0. For $x \neq x_0$, the map $f_x : V_x \to V_x$ is the identity. The map $f_{x_0} : V_{x_0} \to V_{x_0} \otimes W$ is given by $\mathrm{id} \otimes w$. The map f induces a map $(\Phi T_n)(V, L) \to (\Phi T_n)(V \amalg V', L \amalg L')$, which one easily verifies is the map induced by multiplication by $w^{\otimes n}$ on W_n. Thus if v is an element of $(\Phi T_n)(V, L)$, then its product with $w^{\otimes n}$ in W_n can be computed by taking its image under $(\Phi T_n)(f)$. This shows that the product of v and v' belongs to the Δ-module generated by v, which completes the proof. □

Theorem 2.10. *The Δ-module $\Phi(T_n)$ is noetherian.*

Proof. The algebra W_n is noetherian by Theorem 2.6, and so $W_n^{S_n}$ is noetherian by Proposition 2.7. As W_n is a finite $W_n^{S_n}$-module, it is a noetherian $W_n^{S_n}$-module. The same holds for the submodule $\Phi(T_n)$. If M_i is an ascending chain of Δ-submodules of $\Phi(T_n)$ then it is an ascending chain of $W_n^{S_n}$-submodules by the previous proposition, and therefore stabilizes. Thus $\Phi(T_n)$ is a noetherian Δ-module. □

Call a Δ-module *small* if it is a subquotient of a finite direct sum of $\Phi(T_n)$'s (with n allowed to vary). The above theorem implies that small Δ-modules are noetherian, and in particular finitely generated. We record the following result, which follows immediately from the definitions and Proposition 2.9.

Proposition 2.11. *Let F be a small Δ-module. Then there exists a finite chain $0 = F_0 \subset \cdots \subset F_r = F$ of Δ-submodules of F and integers n_i such that F_i/F_{i-1}, regarded as an object of $\mathrm{Sym}(\mathcal{S})$, can be given the structure of a finitely generated module over $W_{n_i}^{S_{n_i}}$.*

Remark 2.12. We can in fact show that all finitely generated Δ-modules are noetherian. The argument in the general case is by a Gröbner degeneration, and is much different than our argument for $\Phi(T_n)$ presented above. However, the above argument for $\Phi(T_n)$, which relates Δ-submodules to modules over $W_n^{S_n}$, is important for our later arguments with Hilbert series.

2.6. **More on Δ-modules.** Let F be a Δ-module. We define $F^{\mathrm{old}}(V, L)$ to be the space spanned by the images of the maps $F(V, \mathscr{U}) \to F(V, L)$, as $\mathscr{U}$ varies over all non-discrete partitions of L. One easily verifies that F^{old} is a Δ-submodule of F. We define a functor

$$\Psi : \mathrm{Mod}_\Delta \to \mathrm{Sym}(\mathcal{S}), \qquad \Psi(F) = F/F^{\mathrm{old}}.$$

Note that $\Psi(F)$ is naturally a Δ-module. However, if $(V, L) \to (V', L')$ is a map in Vec^Δ and $L' \to L$ is not an isomorphism, then $\Psi(F)$ applied to this map is zero; this is why we regard $\Psi(F)$ as an object of $\mathrm{Sym}(\mathcal{S})$. In fact, $\Psi(F)$ is the universal quotient of F with this property. One may thus regard $\Psi(F)$ as the maximal semi-simple quotient (i.e., cosocle) of F.

A Δ-module M is finitely generated if and only if $\Psi(M)$ is a finite length object of $\mathrm{Sym}(\mathcal{S})$; this is a version of Nakayama's lemma. In fact, M is a quotient of $\Phi(\Psi(M))$, though non-canonically. We have $\Psi(\Phi(F)) = F$. The functor Ψ is right exact, but not exact. Its left derived functors $L^i\Psi$ exist. If M is a finitely generated Δ-module then $L^i\Psi(M)$ is a finite length object of $\mathrm{Sym}(\mathcal{S})$; this can be deduced easily from the fact that finitely generated Δ-modules are noetherian. One can recover $[M]$ from $[L\Psi M]$ by applying Φ. Thus the sequence of polynomials $[L^i\Psi M]$ contains more information than the series $[M]$.

We now give an alternative definition of Δ-modules. Recall that for an object (V, L) of Vec^f and a partition $\mathscr{U}$ of L, there is a natural map $(V, \mathscr{U}) \to (V, L)$ in Vec^Δ. One easily verifies that every map in Vec^Δ can be factored as one of these maps followed by a map in Vec^f. In fact, we can even say a bit more. Call a partition *little* if all its parts are singletons, except one which has two elements. Call a map $(V, \mathscr{U}) \to (V, L)$ *little* if $\mathscr{U}$ is. One then verifies that any map $(V, \mathscr{U}) \to (V, L)$ can be factored into a sequence of little maps. Thus all morphisms in Vec^Δ can be factored into little maps and maps in Vec^f. Therefore, a Δ-module can be thought of as an object of $\mathrm{Sym}(\mathcal{S})$ together with the extra data of functoriality with respect to little maps.

This extra data can be recorded in an elegant manner. Let $m^* : \mathcal{S} \to \mathcal{S}^{\otimes 2}$ be the co-multiplication map. It takes $F \in \mathcal{S}$ to the functor $m^*F \in \mathcal{S}^{\otimes 2}$ given by $(V, W) \mapsto F(V \otimes W)$. The functor m^*F has a natural S_2-equivariant structure and so defines an object of $\mathrm{Sym}^2(\mathcal{S})$. There is a unique extension of m^* to a derivation

$$\Delta : \mathrm{Sym}(\mathcal{S}) \to \mathrm{Sym}(\mathcal{S}).$$

Here by "derivation" we mean Δ satisfies the Leibniz rule and interacts correctly with divided powers (Schur functors). A Δ-module is then just an object M of $\mathrm{Sym}(\mathcal{S})$ together with a map $\Delta M \to M$ which satisfies a certain associativity condition, which we do not write out. The map $\Delta M \to M$ precisely records the functoriality of M with respect to little maps in Vec^Δ, and the associativity condition ensures that M extends to a functor with respect to all maps in Vec^Δ. The image of the map $\Delta M \to M$ is M^{old}, and so its cokernel is $\Psi(M)$.

There is an analogy between Δ-modules and graded $\mathbf{C}[t]$-modules. The category $\mathrm{Sym}(\mathcal{S})$ is analogous to the category of graded vector spaces. The functor Φ is analogous to the functor which takes a graded vector space V to the free graded $\mathbf{C}[t]$-module $\mathbf{C}[t] \otimes V$, while the functor Ψ is analogous to the functor which takes a graded $\mathbf{C}[t]$-module M to the graded vector space $M \otimes_{\mathbf{C}[t]} \mathbf{C}$. The map $\Delta M \to M$ is analogous to multiplication by t, while the space M^{old} is analogous to the image of t. One might hope that $M \mapsto L\Psi(M)$ provides an equivalence between the derived category of Mod_Δ and some other natural derived category, in analogy with Koszul duality; we have not worked this out.

3. Hilbert series

In this section we develop the theory of Hilbert series for certain objects of $\mathrm{Sym}(\mathrm{Vec})$, $\mathrm{Sym}(\mathcal{S})$ and Mod_Δ. The main results are rationality theorems. If A is a finitely generated graded ring, in the usual sense, one can prove the rationality of its Hilbert series by picking a surjection $P \to A$, where P is a polynomial ring, resolving A by free P-modules and then explicitly computing the Hilbert series of a free P-module. The key fact that makes this work is that P has finite global dimension. In the setting of twisted commutative algebras, this approach is no longer viable: the twisted commutative algebra $\mathrm{Sym}(U\langle 1 \rangle)$ has infinite global dimension for any non-zero U. The reason for this is that no wedge power of $U\langle 1 \rangle$ vanishes, so the Koszul complex does not terminate! We get around this problem by relating Hilbert series of twisted commutative algebras to equivariant Hilbert series of usual rings, where we can use the usual methods. To study Hilbert series of objects in $\mathrm{Sym}(\mathcal{S})$, we relate them to Hilbert series of twisted commutative algebras. Finally, to study Hilbert series of objects in Mod_Δ (what we ultimately care about), we relate them to Hilbert series of objects in $\mathrm{Sym}(\mathcal{S})$.

3.1. **Hilbert series in** $\mathrm{Sym}(\mathrm{Vec})$. Let M be an object of $\mathrm{Sym}(\mathrm{Vec})$, taken in the sequence model. We assume each M_n is finite dimensional. We define the *Hilbert series* H_M of M by:

$$H_M(t) = \sum_{n=0}^{\infty} \frac{1}{n!} (\dim M_n)\, t^n.$$

Of course, $H_M(t)$ is simply the element $[M]$ of $\mathrm{Sym}(K(\mathrm{Vec})) = \mathbf{Q}[\![t]\!]$. Our goal in this section is to demonstrate the following theorem:

Theorem 3.1. *Let A be a twisted commutative algebra finitely generated in order 1 and let M be a finitely generated A-module. Then $H_M(t)$ is a polynomial in t and e^t.*

Define $H_M^*(t)$ similarly to $H_M(t)$, but without the factorial. The theorem is equivalent to the following statement, which is what we actually prove: we have

$$H_M^*(t) = \sum_{k=0}^{d} \frac{p_k(t)}{(1-kt)^{a_k}}$$

for some integer d, polynomials $p_k(t)$ and non-negative integers a_k. Note that for a module over a graded ring, in the usual sense, the Hilbert series only has a pole at $t = 1$, while our Hilbert series for modules over twisted commutative algebras can have poles at $t = 1/k$ for any non-negative integer k.

The above theorem only applies to modules over twisted commutative algebras generated in order 1, and is false more generally. For example, let $M = A = \mathrm{Sym}((\mathbf{C}^\infty)^{\otimes 2})$, a twisted commutative algebra generated in order 2. Then $H_M(t) = e^{t^2}$. Although this is not a polynomial in t and e^t, it is a very reasonable function, and one can hope that there is a nice generalization of Theorem 3.1.

Before getting into the proof of Theorem 3.1 we introduce equivariant Hilbert series. Say a group G acts on an object M of $\mathrm{Sym}(\mathrm{Vec})$. We define its *$G$-equivariant Hilbert series* $H_{M,G}^*$ by:

$$H_{M,G}^*(t) = \sum_{n=0}^{\infty} [M_n] t^n$$

where $[M_n]$ denotes the class of M_n in the Grothendieck group $K(G)$. Thus $H_{M,G}^*$ is a power series with coefficients in the ring $K(G)$. We will need to use these Hilbert series in our proof of Theorem 3.1 and we will also need a generalization of Theorem 3.1 to the equivariant setting.

We now begin the proof of Theorem 3.1. Thus let A and M be given. Since A is finitely generated in order one, it is a quotient of $\mathrm{Sym}(U\langle 1\rangle)$ for some finite dimensional vector space U. Of course, M is a finitely generated module over $\mathrm{Sym}(U\langle 1\rangle)$. It thus suffices to consider the case where $A = \mathrm{Sym}(U\langle 1\rangle)$. Now, regard A and M in the Schur model. If $\mathbf{S}_\lambda$ occurs in A then λ has at most $\dim U$ rows. Since M is finitely generated, it is a quotient of $A \otimes S$ for some finite length object S of $\mathcal{S}$; it follows that there is an integer d such that only those $\mathbf{S}_\lambda$ for which λ has at most d rows appear in M. We therefore do not lose information by considering $M(\mathbf{C}^d)$ with its $\mathrm{GL}(d)$ action. In fact, we can even consider $M(\mathbf{C}^d)$ with its T action without losing information, where T is the diagonal torus in $\mathrm{GL}(d)$. The main idea of the proof of Theorem 3.1 is to relate H_M^* to $H_{M(\mathbf{C}^d),T}^*$, prove that the latter is of a specific form and then deduce from this the rationality of H_M^*. (One can regard any graded $\mathbf{C}$-algebra as a twisted commutative algebra. Thus $H_{M(\mathbf{C}^d),T}^*$ makes sense. In fact, it agrees with the usual T-equivariant Hilbert series of $M(\mathbf{C}^d)$.)

We need to introduce a bit of notation related to T. We let $\alpha_1, \ldots, \alpha_d$ be the standard projections $T \to \mathbf{G}_m$. We define an involution of the coordinate ring of T, denoted with an overline, by $\overline{\alpha}_i = \alpha_i^{-1}$, and we write $|x|^2$ for $x\overline{x}$. We let $\Delta(\alpha)$ be the discriminant $\prod_{i<j}(\alpha_i - \alpha_j)$. For a character χ of T we define $\int_T \chi(\alpha)d\alpha$ to be 1 if χ is trivial and 0 otherwise, and we extend $\int_T d\alpha$ linearly to all functions on T. (The symbol $\int_T d\alpha$ is just notation and does not indicate actual integration.)

If χ_1 and χ_2 are characters of irreducible representations of $\mathrm{GL}(d)$ then Weyl's integration formula (see [FH, §26.2]), stated in our language, reads

$$\frac{1}{d!} \int_T \chi_1(\alpha)\chi_2(\overline{\alpha})|\Delta(\alpha)|^2 d\alpha = \begin{cases} 1 & \text{if } \chi_1 = \chi_2 \\ 0 & \text{if } \chi_1 \neq \chi_2 \end{cases}$$

We identify $K(T)$ with $\mathbf{Q}[\alpha_i, \alpha_i^{-1}]$ so that a T-equivariant Hilbert series can be identified with a power series in t whose coefficients are Laurent polynomials in the α_i. The following is the key step in our understanding of H_M^*:

Lemma 3.2. *We have*

$$H_M^*(t) = \frac{1}{d!} \int_T H_{M(\mathbf{C}^d),T}^*(t;\alpha) \frac{|\Delta(\alpha)|^2}{1 - \sum \overline{\alpha}_i} d\alpha.$$

Proof. Write $M = \bigoplus \mathbf{S}_\lambda^{\oplus m_\lambda}$, the sum taken over λ. We then have:

$$H_M^*(t) = \sum_\lambda m_\lambda \cdot (\dim \mathbf{M}_\lambda) \cdot t^{|\lambda|}.$$

On the other hand

$$H_{M(\mathbf{C}^d),T}^*(t;\alpha) = \sum_\lambda m_\lambda \cdot (\text{the character of } \mathbf{S}_\lambda(\mathbf{C}^d)) \cdot t^{|\lambda|}.$$

Put

$$f(\alpha) = \sum_\lambda (\text{the character of } \mathbf{S}_\lambda(\mathbf{C}^d)) \cdot \dim \mathbf{M}_\lambda.$$

Weyl's integration formula now gives us

$$H_M^*(t) = \frac{1}{d!} \int_{T'} H_{M(\mathbf{C}^d),T}^*(t;\alpha) f(\overline{\alpha})|\Delta(\alpha)|^2 d\alpha.$$

We must compute $f(\alpha)$. Observe:

$$\bigoplus_{k=0}^{\infty} (\mathbf{C}^d)^{\otimes k} = \bigoplus_\lambda \mathbf{S}_\lambda(\mathbf{C}^d) \otimes \mathbf{M}_\lambda.$$

The character of the right side is $f(\alpha)$. The character of the left side is

$$\sum_{k=0}^{\infty} \left(\sum_{i=1}^{d} \alpha_i \right)^k = \frac{1}{1 - \sum \alpha_i}.$$

This yields the stated formula.   □

We have thus related H_M^*, what we care about, to $H_{M(\mathbf{C}^d),T}^*$, which should be easier to understand since $M(\mathbf{C}^d)$ is a finitely generated module over the polynomial ring $A(\mathbf{C}^d)$. We now see that $H_{M(\mathbf{C}^d),T}^*$ is indeed easy to understand:

Lemma 3.3. *We have*

$$H_{M(\mathbf{C}^d),T}^*(t;\alpha) = \frac{p(t;\alpha)}{\prod_{i=1}^{d}(1 - \alpha_i t)^n}$$

where $p(t;\alpha)$ is a polynomial and $n = \dim U$.

Proof. The terms of the minimal resolution for $M(\mathbf{C}^d)$ over $A(\mathbf{C}^d)$ are $A(\mathbf{C}^d) \otimes E_\bullet$ where

$$E_i = \mathrm{Tor}_i^{A(\mathbf{C}^d)}(M(\mathbf{C}^d), \mathbf{C}).$$

Since Tor is functorial, each E_i carries an action of $\mathrm{GL}(d)$ (and therefore T), and the minimal resolution is equivariant (or rather, can be taken to be so). Thus the T-equivariant Hilbert series

for M is the alternating sum of those for $A(\mathbf{C}^d) \otimes E_i$; of course, each of these is the product of those for $A(\mathbf{C}^d)$ and E_i. Since E_i is a finite dimensional representation of T its Hilbert series is a polynomial. Thus the lemma is reduced to the case $M = A$. Now,

$$A(\mathbf{C}^d) = \mathrm{Sym}(U \otimes \mathbf{C}^d) = \mathrm{Sym}(\mathbf{C}^d \oplus \cdots \oplus \mathbf{C}^d) = \mathrm{Sym}(\mathbf{C}^d) \otimes \cdots \otimes \mathrm{Sym}(\mathbf{C}^d)$$

where $\mathbf{C}^d$ is summed with itself $n = \dim U$ times. We thus find that $H_{A(\mathbf{C}^d),T}$ is the nth power of $H^*_{\mathrm{Sym}(\mathbf{C}^d),T}$. Similarly, $\mathrm{Sym}(\mathbf{C}^d) = \mathrm{Sym}(\mathbf{C} \oplus \cdots \oplus \mathbf{C})$, where there are d copies of $\mathbf{C}$ and T acts on the ith one by the character α_i. Thus $H^*_{\mathrm{Sym}(\mathbf{C}^d),T} = \prod(1 - \alpha_i t)^{-1}$. This proves the lemma. $\square$

Combining the two lemmas, we obtain an expression

$$H^*_M(t) = \int_T \frac{p(t; \alpha)}{\prod(1 - \alpha_i t)^n} \frac{1}{1 - \sum \overline{\alpha}_i} d\alpha$$

where $p(t; \alpha)$ is a polynomial in t, the α_i and the α_i^{-1}. (We have absorbed the $1/d!$ and $|\Delta(\alpha)|^2$ into p.) Expanding the integrand into a power series, we find

$$H^*_M(t) = \int_T \left[\sum_{k,\ell} \begin{bmatrix} k \\ n \end{bmatrix} \alpha^k \left(\sum \overline{\alpha}_i \right)^\ell p(t; \alpha) t^{|k|} \right] d\alpha$$

where the sum is taken over $k \in \mathbf{Z}^d_{\geq 0}$ and $\ell \in \mathbf{Z}_{\geq 0}$. Here

$$\begin{bmatrix} k \\ n \end{bmatrix} = \binom{k_1 + n - 1}{n - 1} \cdots \binom{k_d + n - 1}{n - 1}, \qquad \alpha^k = \alpha_1^{k_1} \cdots \alpha_d^{k_d} \qquad \text{and} \qquad |k| = k_1 + \cdots + k_d.$$

We must show that this is a rational function in t. It suffices, by linearity, to treat the case where $p(t; \alpha) = t^{e_0} \alpha_1^{e_1} \cdots \alpha_d^{e_d}$ where the e_i are integers. Of course, the t^{e_0} factor does not really affect anything, so we leave it out. We are thus reduced to showing that

$$\int_T \left[\sum_{k,\ell} \begin{bmatrix} k \\ n \end{bmatrix} \alpha^{k+e} \left(\sum \overline{\alpha}_i \right)^\ell t^{|k|} \right] d\alpha$$

is rational. By degree considerations, the (k, ℓ) term in the above sum integrates to zero unless $\ell = |k + e|$. Furthermore, when $\ell = |k + e|$ only one monomial in $(\sum \overline{\alpha}_i)^\ell$ contributes a non-zero quantity, namely the one where $\overline{\alpha}_i$ has exponent $k_i + e_i$. We therefore find that the above is equal to

$$\sum_k \begin{bmatrix} k \\ n \end{bmatrix} C_{k+e} t^{|k|}$$

where

$$C_k = \frac{(|k|)!}{(k_1)! \cdots (k_d)!}$$

is the multinomial coefficient. (We use the convention that $C_k = 0$ if any of the coordinates of k are negative.) Theorem 3.1 now follows from the following lemma:

Lemma 3.4. *Let d be a positive integer, let $e \in \mathbf{Z}^d$ and let p be a polynomial of d variables. Then*

$$\sum p(k) C_{k+e} t^{|k|}$$

is a rational function of t, with poles only at $t = 1/a$ where $1 \leq a \leq d$ is an integer. (The sum is taken over $k \in \mathbf{Z}^d_{\geq 0}$.)

Proof. Observe that the formula

$$k_1 C_{k_1,k_2,\ldots,k_d} = |k| C_{k_1-1,k_2,\ldots,k_d}$$

is valid for any tuple of integers $k \in \mathbf{Z}^d$. To prove the lemma, it suffices to treat the case where p is a monomial. Thus, if p is not a constant, we can write $p = k_i p'$ for some index i and some smaller monomial p'. We therefore have

$$\sum p(k) C_{k+e} t^{|k|} = \sum k_i p'(k) C_{k+e} t^{|k|}$$
$$= \sum p'(k)(k_i + e_i - e_i) C_{k+e} t^{|k|}$$
$$= \sum p'(k) |k+e| C_{k+e'} t^{|k|} - e_i \sum p'(k) C_{k+e} t^{|k|}$$

where e' is obtained from e by replacing e_i with $e_i - 1$. In the right term we have replaced p with a lower degree polynomial. In the left term we have replaced p with an equal degree polynomial, but one that is of the form $p'(k)|k|$ where p' has smaller degree. (Note that $|k + e| = |k| + |e|$.) It follows that by repeatedly applying the above process, we can reduce to the case where p is a function of $|k|$. Now, note that

$$\sum |k|^n C_{k+e} t^{|k|} = \left(t \frac{d}{dt} \right)^n \sum C_{k+e} t^{|k|}.$$

It thus suffices to show that $\sum C_{k+e} t^{|k|}$ is a rational function. We have

$$\sum_{k \geq 0} C_{k+e} t^{|k|} = \sum_{k \geq e} C_k t^{|k-e|} = t^{-|e|} \sum_{k \geq e} C_k t^{|k|}$$

where $k \geq e$ means $k_i \geq e_i$ for each i. Now, the terms in the right sum for which some k_i is negative are zero and therefore do not contribute. We can thus assume that each e_i is non-negative. We can also ignore the $t^{-|e|}$ factor. Now, write $k = (k_1, k')$ where k' is a $d - 1$ tuple, and do similarly for e. Then

$$\sum_{k \geq e} C_k t^{|k|} = \sum_{k_1 \geq e_1, k' \geq e'} C_k t^{|k|} = \sum_{k_1 \geq 0, k' \geq e'} C_k t^{|k|} - \sum_{k_1=0}^{e_1-1} \sum_{k' \geq e'} C_k t^{|k|}$$

Now, for k_1 fixed, we have

$$\sum_{k' \geq e'} C_k t^{|k|} = \frac{t^{k_1}}{k_1!} \sum_{k' \geq e'} (|k'| + k_1) \cdots (|k'| + 1) C_{k'} t^{|k'|}.$$

Thus each of the sums on the right in the previous expression is of the general form that we are considering in this lemma but with a smaller d. We can therefore assume that they are each rational by induction. By repeating this procedure, we can thus reduce to the case $e = 0$. The identity

$$\sum_{|k|=n} C_k = d^n$$

now gives

$$\sum_{k \geq 0} C_k t^{|k|} = \frac{1}{1 - dt}.$$

This completes the proof. $\qquad\qquad\square$

3.2. **Equivariant Hilbert series in** Sym(Vec). Unfortunately, Theorem 3.1 is too weak for our eventual applications. Before stating the result we need, we make a definition for the sake of clarity:

Definition 3.5. Let A be a ring. A series $f \in A[\![t]\!]$ is *rational* if there exists a polynomial $q \in A[t]$ with $q(0) = 1$ such that qf is a polynomial.

Note that it could be that $f \in A[\![t]\!]$ is not rational, but that there is an extension $A \subset B$ so that f is rational when regarded as an element of $B[\![t]\!]$. Thus a bit of care needs to be taken with the definition. However, we do have the following simple result, the proof of which is left to the reader:

Lemma 3.6. *Let $A \subset B$ be an inclusion of rings and let f be an element of $A[\![t]\!]$ such that f is rational when regarded as an element of $B[\![t]\!]$. Then f itself is rational in the following cases: (1) there is a finite group G acting on B such that $A = B^G$; (2) A and B are fields.*

The main result of this section is the following:

Theorem 3.7. *Let G be a connected reductive group, let Γ be a finite group, let A be a twisted commutative algebra finitely generated in order 1 on which $G \times \Gamma$ acts and let M be a finitely generated A-module with a compatible action of $G \times \Gamma$. Then the G-equivariant Hilbert series $H^*_{M^\Gamma, G}$ of M^Γ, regarded as an element of the power series ring $K(G)[\![t]\!]$, is a rational function.*

Most likely, this proposition could be generalized by replacing $\Gamma \subset G \times \Gamma$ with an arbitrary normal reductive subgroup of an arbitrary reductive group. We do not need this more general result, and so only prove the special one, which allows for some simplifications in the proof. With some book-keeping, one can also show that the denominator of $H^*_{M^\Gamma, G}$ has a particular form, but we do not do this. The theorem implies a certain result about $H_{M^\Gamma, G}$, but one that is not so elegant: exponentials of algebraic functions (roots of polynomials over $K(G)$) appear. We prove this proposition following the same plan as the proof of last one, after some preliminary reductions.

First, we observe that it suffices to prove that $H^*_{M, \Gamma \times G}$, an element of $K(\Gamma \times G)[\![t]\!]$, is a rational function. To see this, assume we have an equation $(1+tq)H^*_{M, \Gamma \times G} = p$ with p and q in $K(\Gamma \times G)[\![t]\!]$. Now, observe that $K(\Gamma)$ can be thought of as the ring of class functions on Γ (at least, after an extension of scalars, which does not affect rationality by Lemma 3.6) and so we may write

$$H^*_{M, \Gamma \times G} = \sum H_i \delta_i, \qquad q = \sum q_i \delta_i, \qquad p = \sum p_i \delta_i$$

where H_i, q_i and t_i belong to $K(G)[\![t]\!]$ and the δ_i are characteristic functions of conjugacy classes in Γ. Since the δ_i are orthogonal idempotents, we have

$$(1 + tq)H^*_{M, \Gamma \times G} = \sum (1 + tq_i)H_i \delta_i = \sum p_i \delta_i$$

and so $(1 + tq_i)H_i = p_i$ holds for each i. Thus each H_i is a rational function in $K(G)[\![t]\!]$. Since $H^*_{M^\Gamma, G} = (\#\Gamma)^{-1} \sum H_i$, it follows that it too is a rational function. This establishes our claim.

Now, we can think of the coefficients of $H^*_{M, \Gamma \times G}$ as class functions on Γ. The series $H^*_{M, \Gamma \times G}$ defines a rational element of $K(\Gamma \times G)[\![t]\!]$ if and only if the series $H^*_{M, \gamma, G}$ obtained by evaluating on the element $\gamma \in \Gamma$ is a rational element of $K(G)[\![t]\!]$, for each γ. Let Γ' be the cyclic subgroup generated by some $\gamma \in \Gamma$. By the same reasoning, $H^*_{M, \gamma, G}$ will be a rational element of $K(G)[\![t]\!]$ if $H^*_{M, \Gamma' \times G}$ is a rational element of $K(\Gamma' \times G)[\![t]\!]$. Thus it suffices to show that for each cyclic subgroup Γ' of Γ, the series $H^*_{M\Gamma' \times G}$ is a rational element of $K(\Gamma' \times G)[\![t]\!]$. In other words, we may assume from the outset that Γ is cyclic.

We make another reduction. We have $K(G) = K(H)^W$ where H is a maximal torus in G and W is its Weyl group. By Lemma 3.6, a power series with coefficients in $K(G)$ is rational if and only if it is so when regarded with $K(H)$ coefficients. Thus it suffices to show that $H^*_{M, \Gamma \times H}$ is rational. In other words, we may as well assume from the outset that G is a torus.

Finally, as in the proof of the Theorem 3.1, we may assume $A = \mathrm{Sym}(U\langle 1\rangle)$, where U is a finite dimensional representation of $\Gamma \times G$. Since $\Gamma \times G$ is a commutative reductive group, we can write

$U = \bigoplus_{j=1}^{n} \mathbf{C}\psi_j$ where the ψ_j are characters of $\Gamma \times G$. Pick d large compared to the number of rows appearing in M and let $T \subset \mathrm{GL}(d)$ be the diagonal torus. We now have:

Lemma 3.8. *We have*

$$H^*_{M(\mathbf{C}^d),\Gamma \times G \times T}(t;\alpha) = \frac{p(t;\alpha)}{\prod_{ij}(1 - \alpha_i\psi_j t)}.$$

Here p belongs to $K(\Gamma \times G \times T)[t] = K(\Gamma \times G)[t, \alpha_i]$. The product is taken over $1 \leq i \leq d$ and $1 \leq j \leq \dim U$.

Proof. As before, we can reduce to the case $M = A$ by considering the minimal resolution of M. An easy computation, similar to the previous one, gives $H^*_{A(\mathbf{C}^d),\Gamma \times G \times T} = \prod_{ij}(1 - \alpha_i\psi_j t)^{-1}$. $\square$

Lemma 3.2 carries over exactly to the present situation. Combining it with the previous lemma yields

(5) $$H^*_{M,\Gamma \times G}(t) = \int_T \frac{p(t;\alpha)}{\prod_{ij}(1 - \alpha_i\psi_j t)} \frac{1}{1 - (\sum \overline{\alpha_i})} d\alpha.$$

For i fixed, we have

$$\frac{1}{\prod_j(1 - \alpha_i\psi_j t)} = \sum \alpha_i^{|a|}\psi^a t^{|a|}$$

where the sum is taken over $a \in \mathbf{Z}^n_{\geq 0}$, with $n = \dim U$, $|a|$ is defined as $a_1 + \cdots + a_n$ and ψ^a is defined as $\psi_1^{a_1} \cdots \psi_n^{a_n}$. Define

$$\begin{bmatrix} k \\ n \end{bmatrix}_\psi = \sum_{|a|=k} \psi^a,$$

where a belongs to $\mathbf{Z}^n_{\geq 0}$, so that

$$\frac{1}{\prod_j(1 - \alpha_i\psi_j t)} = \sum_{k=0}^{\infty} \begin{bmatrix} k \\ n \end{bmatrix}_\psi \alpha_i^k t^k.$$

With this notation in hand, we now expand the integrand of (5) into a series. We find

$$H^*_{M,\Gamma \times G}(t) = \int_T \left[\sum_{k,\ell} \begin{bmatrix} k \\ n \end{bmatrix}_\psi \alpha^k \left(\sum \overline{\alpha_i}\right)^\ell p(t;\alpha)t^{|k|} \right] d\alpha.$$

The sum is taken over $k \in \mathbf{Z}^d_{\geq 0}$ and $\ell \in \mathbf{Z}_{\geq 0}$ and our notation is as before; we mention

$$\begin{bmatrix} k \\ n \end{bmatrix}_\psi = \begin{bmatrix} k_1 \\ n \end{bmatrix}_\psi \cdots \begin{bmatrix} k_d \\ n \end{bmatrix}_\psi.$$

We must show that the previous equation is rational in t. By linearity, it suffices to treat the case where $p(t;\alpha)$ is of the form $xt^{e_0}\alpha_1^{e_1} \cdots \alpha_d^{e_d}$ where x belongs to $K(\Gamma \times G)$ and the e_i are integers. Of course, xt^{e_0} pulls out of the integral, and can thus be safely ignored. Hence, it suffices to consider the case where p is α^e, with $e = (e_1, \ldots, e_d)$. As before, the (k,ℓ) term only contributes if $\ell = |k+e|$ and then only one term of $(\sum \overline{\alpha_i})^\ell$ contributes. The previous integral thus evaluates to:

$$\sum_k \begin{bmatrix} k \\ n \end{bmatrix}_\psi C_{k+e} t^{|k|}.$$

Now, for a single integer k the expression $\begin{bmatrix} k \\ n \end{bmatrix}_\psi$ is of the form $\sum_i a_i\psi_i^k$ where a_i is a rational function of the ψ. It follows that for $k \in \mathbf{Z}^d_{\geq 0}$ the expression $\begin{bmatrix} k \\ n \end{bmatrix}_\psi$ is of the form $\sum_i a_i\psi_i^k$ where

the sum is taken over tuples $i \in \{1, \ldots, n\}^d$ and ψ_i^k denotes $\psi_{i_1}^{k_1} \cdots \psi_{i_d}^{k_d}$; again the a_i are rational functions in the ψ. It thus suffices to show that for each such tuple i, the expression

$$\sum_k \psi_i^k C_{k+e} t^{|k|}$$

is rational in t. This is accomplished in the following lemma, which completes the proof of Theorem 3.7:

Lemma 3.9. *Keep the above notation and let p be a polynomial. Then*

$$\sum p(k) \psi_i^k C_{k+e} t^{|k|}$$

is a rational function of t. (The sum is taken over $k \in \mathbf{Z}_{\geq 0}^d$.) Furthermore, the only denominators which appear are of the form $1 - at$ where a is a sum of at most d of the ψ's.

Proof. As in the proof of Lemma 3.4, we reduce to the case where $p = 1$. We then change k to $k - e$ and pull monomials out of the sum, to obtain an expression of the form

$$\sum_{k \geq e} \psi_i^k C_k t^{|k|}.$$

The difference

$$\sum_{k \geq 0} \psi_i^k C_k t^{|k|} - \sum_{k \geq e} \psi_i^k C_k t^{|k|}$$

is a finite sum of sums of the form considered in the lemma, but with a smaller value of d. (The terms which appear may no longer have $p = 1$; this is the reason for including p in the general form of the sum we consider.) It thus suffices to show that the first sum above is rational in t. We have

$$\sum_{|k|=n} \psi_i^k C_k = (\psi_{i_1} + \cdots + \psi_{i_d})^n = a^n$$

by the multinomial theorem. Thus the first sum in the previous expression is $(1 - at)^{-1}$, which completes the proof. □

3.3. **Hilbert series in** $\mathrm{Sym}(\mathcal{S})$.

Let M be an object of $\mathrm{Sym}(\mathcal{S})$. We can regard M as a sequence of equivariant functors $M_n : \mathrm{Vec}^n \to \mathrm{Vec}$. Write

$$M_n(V_1, \ldots, V_n) = \bigoplus_{i \in I_n} \mathbf{S}_{\lambda_{i,1}}(V_1) \otimes \cdots \otimes \mathbf{S}_{\lambda_{i,n}}(V_n)$$

for some index set I_n and partitions $\lambda_{i,j}$ (both depending on n). We assume I_n is finite for each n. Put

$$m_n^* = \sum_{i \in I_n} s_{\lambda_{i,1}} \cdots s_{\lambda_{i,n}}, \qquad H_M^*(t) = \sum_{n=0}^{\infty} m_n^* t^n.$$

We regard m_n^* as an element of the polynomial ring $\mathbf{Q}[s_\lambda]$ and $H_M^*(t)$ as an element of the power series ring $\mathbf{Q}[s_\lambda][\![t]\!]$. The variable t is basically superfluous, since the power of t can be obtained from the order of the polynomial m_n^*. When t is omitted (or set to 1), H_M^* agrees with $[M]^*$. One can also define $H_M(t)$, which is analogous to $[M]$, by replacing m_n^* with $m_n = \frac{1}{n!} m_n^*$. Our main result concerning these series is the following:

Theorem 3.10. *Let A be an algebra in $\mathrm{Sym}(\mathcal{S})$ finitely generated in order 1 on which a finite group Γ acts and let M be a finitely generated A-module with a compatible action of Γ. Then $H_{M^\Gamma}^*$ is a rational function of the s_λ.*

As with Theorem 3.7, this result does not translate to an elegant statement about H_{M^Γ}. We now explain the basic strategy of our proof, in the case where Γ is trivial. Let M be a finitely generated A-module. We would like to relate H_M^* to the Hilbert series of an object of $\mathrm{Sym}(\mathrm{Vec})$, so that we can apply the results we have established in that case. The most obvious way to obtain an object of $\mathrm{Sym}(\mathrm{Vec})$ is to pick a vector space U, let $i : (\mathrm{fs}) \to \mathrm{Vec}^f$ be the functor assigning to L the constant family U_L and then consider i^*M. Unfortunately, H_M^* cannot be recovered from $H_{i^*M}^*$, or even $H_{i^*M,\mathrm{GL}(U)}^*$. Thus the most obvious approach fails. However, a slight modification works: instead of picking just one vector space U we pick finitely many $U_1, \ldots, U_r$ and build from M an object of $\mathrm{Sym}(\mathrm{Vec})$ with an action of $\mathrm{GL}(U_1) \times \cdots \times \mathrm{GL}(U_r)$. It turns out that, due to the form of M, we can take r large enough so that information is not lost — this is essentially the content of Proposition 3.14 below. Before proving that result, we need some lemmas, the first two of which are left to the reader.

Lemma 3.11. *Let $f : \mathbf{C}^n \to \mathbf{C}^m$ be a polynomial map whose components are homogeneous of positive degree and whose image is not contained in any linear subspace of $\mathbf{C}^m$. Then for $r \gg 0$ any element of $\mathbf{C}^m$ can be expressed as a sum of r elements of the image of f.*

Lemma 3.12. *Let K be a field and let (f_i) be a sequence of elements in K. Assume that there exists $m > 0$ such that for all $k_1, \ldots, k_m$ sufficiently large, the $m \times m$ matrix (f_{k_i-j}) has determinant zero. Then $\sum_{i \geq 0} f_i t^i$ can be expressed in the form a/b where a and b belong to $K[t]$ and $\deg b \leq m - 1$.*

Lemma 3.13. *Let A be a UFD and let (f_i) be a sequence of elements of A. Assume that there exists an integer m and elements $\alpha_1, \ldots, \alpha_m$ of A such that*

$$f_k = \sum_{i=1}^{m} \alpha_i f_{k-i}$$

holds for all $k \gg 0$. Amongst all such expressions choose one with m minimal. Then given any N there exist $k_1, \ldots, k_m > N$ such that the determinant of the matrix (f_{k_i-j}) is non-zero.

Proof. Put $f = \sum f_i t^i$, an element of $A[\![t]\!]$, and $q = 1 - \sum_{i=1}^m \alpha_i t^i$, an element of $A[t]$. We have that qf belongs to $A[t]$; write $p = qf$ so that $f = p/q$. Note that $A[t]$ is also a UFD and that the minimality assumption on m implies that p and q are coprime. Indeed, say p and q are both divisible by some non-unit $r \in A[t]$. Then we can write $p = rp'$ and $q = rq'$. Evaluating the second expression at $t = 0$ gives $1 = r(0)q'(0)$ so that $r(0)$ and $q'(0)$ both belong to $A^\times$. We can therefore scale r so that $r(0) = 1$, in which case $q'(0) = 1$ as well. Since r is assumed to be a non-unit and $r(0)$ is a unit, it follows that r has degree at least 1. We then have $f = p'/q'$ with $q'(0) = 1$ and $\deg q' < m$, contradicting the minimality of m.

Now, assume for the sake of contradiction that $\det(f_{k_i-j}) = 0$ for all sufficiently large k_i. By Lemma 3.12 we have $f = g/h$ where g and h belong to $K[t]$ and $\deg h < m$. Here K is the field of fractions of A. Pick $a \in A$ non-zero so that ag and ah belong to $A[t]$. We then have $(ah)p = (ag)q$. Since p is coprime to q it follows that ah is divisible by q. However, this contradicts h having smaller degree than q. We thus conclude that $\det(f_{k_i-j})$ cannot vanish for all sufficiently large k_i. $\square$

Proposition 3.14. *Let W be a finite dimensional vector space over a subfield $\mathbf{C}$ and let V be a finite dimensional subspace of $\mathrm{Sym}(W)$ spanned by homogeneous elements. For a positive integer r, let $i_r : \mathrm{Sym}(V) \to \mathrm{Sym}(W)^{\otimes r}$ be the ring homomorphism which on V is given by $x \mapsto \sum \{x\}_i$, where $\{-\}_i : \mathrm{Sym}(W) \to \mathrm{Sym}(W)^{\otimes r}$ is the ring homomorphism given by inclusion into the ith factor. Then for $r \gg 0$ we have:*

(a) *The map i_r is injective.*
(b) *If $x \in \mathrm{Frac}(\mathrm{Sym}(V))$ and $i_r(x)$ belongs to $\mathrm{Sym}(W)^{\otimes r}$ then x belongs to $\mathrm{Sym}(V)$.*
(c) *A series $f \in \mathrm{Sym}(V)[\![t]\!]$ is rational if and only if $i_r(f)$ is.*

Proof. We can check the conclusions of the proposition by tensoring up to $\mathbf{C}$, and so we may thus assume we are working with complex vector spaces. The map $\mathrm{Sym}(V) \to \mathrm{Sym}(W)$ corresponds to an algebraic map $f : W^* \to V^*$ whose components are homogeneous of positive degree. Since $V \to \mathrm{Sym}(W)$ is injective, the image of f is not contained in any linear subspace of V^*. It thus follows from Lemma 3.11 that for $r \gg 0$, any element of V^* is a sum of r elements of the image of f. Now, the map i_r corresponds to the map $i_r^* : (W^*)^r \to V^*$ given by $(w_1, \ldots, w_n) \mapsto \sum f(w_i)$. We thus see that i_r^* is surjective for all $r \gg 0$.

Fix $r \gg 0$ and put $i = i_r$. We now show that (a) and (b) hold. The equation $i(x) = 0$ is equivalent to $x \circ i^* = 0$, where x is thought of as a function $V^* \to \mathbf{C}$. Since i^* is surjective, this equation implies $x = 0$. This shows that i is injective. Now say that x belongs to $\mathrm{Frac}(\mathrm{Sym}(V))$ and $i(x)$ is a polynomial. Then x defines a rational function on V^* such that $x \circ i^*$ is a regular function on $(W^*)^r$. Since i^* is surjective, x must be regular on V^* and thus it belongs to $\mathrm{Sym}(V)$.

We now prove (c). It is clear that if f is rational then $i(f)$ is as well. Thus let $f \in \mathrm{Sym}(V)[\![t]\!]$ be given and assume that $i(f)$ is rational. Write $f = \sum f_i t^i$ with $f_i \in \mathrm{Sym}(V)$. The rationality of $i(f)$ means that we can find a polynomial $q = 1 - \sum_{i=1}^m \alpha_i t^i$ with $\alpha_i \in \mathrm{Sym}(W)^{\otimes r}$ such that qf is a polynomial. Choose q with m minimal. We then have

$$f_k = \sum_{i=1}^m \alpha_i f_{k-i}$$

for all sufficiently large k. Thus for all large $k_1, \ldots, k_m$ we have the equation $Ax = y$ where A is the $m \times m$ matrix $(f_{k_i - j})$, x is the column vector (α_i) and y is the column vector (f_{k_i}). By the minimality of m and Lemma 3.13 we can pick k_i so that $\det A$ is non-zero. We then find $x = A^{-1}y$, which shows that α_i belongs to $i(\mathrm{Frac}(\mathrm{Sym}(V)))$. Since α_i also belongs to $\mathrm{Sym}(W)^{\otimes r}$, statement (b) of the lemma implies that each α_i belongs to $i(\mathrm{Sym}(V))$. Thus $q = i(q')$ for a unique q' in $\mathrm{Sym}(V)[t]$ with the required properties to establish that f is rational. $\qquad\square$

We now prove the theorem:

Proof of Theorem 3.10. Let Γ, A and M be given. Let P be the set of partitions appearing in M. The set P is finite; indeed, only finitely many partitions appear in A (see the discussion preceding Theorem 2.6) and M is a quotient of $A \otimes F$ for some finite length object F of $\mathrm{Sym}(\mathcal{S})$. Let V be the subspace of $K(\mathcal{S})$ spanned by the s_λ with $\lambda \in P$. Thus $H_{M^\Gamma}^*$ belongs to $\mathrm{Sym}(V)[\![t]\!]$.

Let U be a finite dimensional vector space whose dimension exceeds the number of rows of any partition appearing in P and let $G = \mathrm{SL}(U)$. Then $K(G)$ is a polynomial ring. (We use $\mathrm{SL}(U)$ instead of $\mathrm{GL}(U)$ so that the Grothendieck group is a polynomial ring; it makes the argument a bit cleaner.) Evaluation on U gives a map $K(\mathcal{S}) \to K(G)$ which is injective when restricted to V. Let r be a large integer and let

$$\phi : \mathrm{Sym}(K(\mathcal{S})) \to K(G)^{\otimes r}$$

be the ring map which is given by

$$\phi([S]) = \sum_{i=1}^r \{[S(U)]\}_i$$

for $[S]$ in $K(\mathcal{S})$, where $\{-\}_i : K(G) \to K(G)^{\otimes r}$ is the inclusion in the ith factor. By Proposition 3.14(c), a power series $f \in \mathrm{Sym}(V)[\![t]\!]$ is a rational function if and only if $\phi(f)$ is. It thus suffices to show that $\phi(H_{M^\Gamma}^*)$ is rational.

To understand the map ϕ we lift it to a functor Φ, as follows. First, identify $K(G)^{\otimes r}$ with $K(G^r)$. Let $U_1, \ldots, U_r$ be copies of U and put $G_i = \mathrm{SL}(U_i)$; we think of G^r as $G_1 \times \cdots \times G_r$. We regard ϕ as a map

$$\phi : \mathrm{Sym}(K(\mathcal{S})) \to K(G_1 \times \cdots \times G_r).$$

It can be described explicitly on $K(\mathcal{S})$ as follows:

$$\phi([S]) = \sum_{i=1}^{r} [S(U_i)].$$

Now define Φ to be the functor

$$\Phi : \mathrm{Sym}(\mathcal{S}) \to \mathrm{Sym}(\mathrm{Vec}), \qquad \Phi(F)(L) = \bigoplus_{L=L_1 \amalg \cdots \amalg L_r} F((U_1)_{L_1} \amalg \cdots \amalg (U_r)_{L_r}).$$

Here the sum is over all partitions of L into r parts and $(U_i)_{L_i}$ denotes the family (V, L_i) where $V_x = U_i$ for all $x \in L_i$. (A more conceptual description of Φ is as follows. Let $i : (\mathrm{fs})^r \to \mathrm{Vec}^f$ take $(L_1, \ldots, L_r)$ to $(U_1)_{L_1} \amalg \cdots \amalg (U_r)_{L_r}$ and let $j : (\mathrm{fs})^r \to (\mathrm{fs})$ be the addition map. Then $\Phi(F) = j_* i^* F$.) One readily verifies that Φ is a tensor functor. We now claim that Φ lifts ϕ, that is, we have

$$\phi([N]) = [\Phi(N)]$$

for all N in $\mathrm{Sym}(\mathcal{S})$. Both sides above are additive in N so it suffices to treat the case where N is a simple object of $\mathrm{Sym}(\mathcal{S})$. As we have previously stated (§2.2), the simple objects are of the form $\bigotimes \mathbf{S}_{\lambda_i}(S_i)$ where the λ_i are partitions and S_i are distinct simple objects of $\mathcal{S}$. Since Φ is a tensor functor and ϕ is a ring homomorphism, we are reduced to the case of considering $N = \mathbf{S}_\lambda(S)$. Put $k = |\lambda|$. Then N is the equivariant functor $\mathrm{Vec}^k \to \mathrm{Vec}$ given by

$$(V_1, \ldots, V_k) \mapsto \mathbf{M}_\lambda \otimes S(V_1) \otimes \cdots \otimes S(V_k).$$

We thus find that $\Phi(N) \in \mathrm{Sym}(\mathrm{Vec})$ is supported in order k and assigns to the set $\{1, \ldots, k\}$ the space

$$\mathbf{M}_\lambda \otimes \bigoplus_{i_1, \ldots, i_k} S(U_{i_1}) \otimes \cdots \otimes S(U_{i_k})$$

where the sum is over all $(i_1, \ldots, i_k)$ in $\{1, \ldots, r\}^k$. We therefore have

$$[\Phi(N)] = \frac{\dim \mathbf{M}_\lambda}{k!} \left(\sum_{i=1}^{r} [S(U_i)] \right)^k.$$

On the other hand,

$$[N] = \frac{\dim \mathbf{M}_\lambda}{k!} [S]^k.$$

Applying ϕ to the above gives exactly the previous formula for $[\Phi(N)]$. This proves the claim.

The above discussion, and the fact that Φ commutes with the formation of Γ invariants, shows that

$$\phi(H^*_{M^\Gamma}) = H^*_{\Phi(M)^\Gamma, G_1 \times \cdots \times G_r}.$$

As Φ is a tensor functor, $\Phi(A)$ is a twisted commutative algebra finitely generated in order 1 and $\Phi(M)$ is a finitely generated module over it. Thus $H^*_{\Phi(M)^\Gamma, G_1 \times \cdots \times G_r}$ is a rational function in $K(G_1 \times \cdots \times G_r)[\![t]\!]$ by Theorem 3.7. We thus find that $\phi(H^*_{M^\Gamma})$ is rational, which completes the proof. $\qquad \square$

We note the following corollary of the proposition.

Corollary 3.15. *Let A be an algebra in $\mathrm{Sym}(\mathcal{S})$ finitely generated in order 1 on which a finite group Γ acts and let M be a finitely generated A^Γ-module. Then H^*_M is a rational function of the s_λ.*

Proof. Let $N = A \otimes_{A^\Gamma} M$. Then N is a finitely generated A-module and $N^\Gamma = M$, where Γ acts on N by acting trivially on M. We thus have that $H^*_M = H^*_{N^\Gamma}$ is rational by the proposition. $\qquad \square$

3.4. **Hilbert series of Δ-modules.** Let M be a Δ-module. We define its Hilbert series to be the Hilbert series of the underlying object of $\mathrm{Sym}(\mathcal{S})$. Our main result on such Hilbert series is the following, which follows immediately from Corollary 3.15 and Proposition 2.11.

Theorem 3.16. *Let M be a small Δ-module. Then H_M^* is a rational function of the s_λ.*

Remark 3.17. We expect this result to hold for all finitely generated Δ-modules. We can prove it for a much larger class than the class of small Δ-modules, but we have not been able to prove it for all finitely generated Δ-modules.

4. Syzygies of Δ-schemes

We now apply the theory we have developed to the study of the syzygies of certain families of schemes, which we call Δ-schemes.

4.1. **Δ-schemes.** An *abstract Δ-scheme* is a functor X from the category Vec^Δ to the category of schemes over $\mathbf{C}$. The notion of a morphism of abstract Δ-schemes is evident. In this way we have a category of abstract Δ-schemes. We say that a morphism $X \to Y$ of abstract Δ-schemes is a closed immersion if $X(V, L) \to Y(V, L)$ is a closed immersion for all (V, L).

The category of abstract Δ-schemes is too large for our purposes; we now introduce a more manageable category. For an object (V, L) of Vec^Δ, let $\mathbf{V}(V, L)$ be the vector space $\bigotimes_{x \in L} V_x^*$, regarded as a scheme. Then $\mathbf{V}$ is an abstract Δ-scheme, in an obvious way. A *Δ-scheme* is a pair (X, i) consisting of an abstract Δ-scheme X such that $X(V, L)$ is non-empty for all (V, L) and a closed immersion $i : X \to \mathbf{V}$. Note that if X is a Δ-scheme, then $X(V, L) \subset \mathbf{V}(V, L)$ is closed under scaling (by functoriality), and is thus a cone. A morphism of Δ-schemes is a morphism of abstract Δ-schemes which commutes with the embeddings into $\mathbf{V}$. There is at most one morphism between two Δ-schemes, and so the category of Δ-schemes is partially ordered.

We can think of Δ-schemes in terms of their ideals of definitions, as follows. Let $P(V, L)$ be the affine coordinate ring of $\mathbf{V}(V, L)$. We have

$$P(V, L) = \mathrm{Sym}\left(\bigotimes_{x \in L} V_x\right).$$

A *Δ-ideal* is a rule which assigns to each object (V, L) of Vec^Δ an ideal $I(V, L)$ of $P(V, L)$ such that if $(V, L) \to (V', L')$ is a morphism in Vec^Δ then under the map $P(V, L) \to P(V', L')$ the ideal $I(V, L)$ maps into $I(V', L')$. Suppose I is a Δ-ideal. Let $X(V, L)$ be the subscheme of $\mathbf{V}(V, L)$ cut out by $I(V, L)$. Then X is naturally a Δ-scheme. The association $I \mapsto X$ is an anti-equivalence of categories. By an *element* of P we mean an element of $P(V, L)$ for some (V, L). Given any collection S of elements of P there is a unique minimal Δ-ideal I containing S; we call I the Δ-ideal *generated* by S.

In more concrete terms, a Δ-scheme can be thought of as a sequence of rules $(X_n)_{n \geq 0}$, where X_n assigns to each n-tuple of vector spaces $(V_1, \ldots, V_n)$ a closed subscheme $X_n(V_1, \ldots, V_n)$ of $\bigotimes V_i^*$, such that: X_n is functorial in $(V_1, \ldots, V_n)$; X_n is S_n-equivariant; and $X_{n+1}(V_1, \ldots, V_{n+1})$ is contained in $X_n(V_1, \ldots, V_{n-1}, V_n \otimes V_{n+1})$. Of course, we can describe Δ-ideals similarly.

4.2. **Examples of Δ-schemes.** We now give some examples of Δ-schemes. There are two rather trivial examples: $\mathbf{V}$ itself is a Δ-scheme, and the final object in the category; and the rule $(V, L) \mapsto \mathrm{Spec}(\mathbf{C})$ (embedded in $\mathbf{V}$ as the origin) is a Δ-scheme, and the initial object in the category. A more interesting example, and the one which motivated the whole theory, is the Segre embedding: take $X(V, L)$ to be the set of pure tensors in $\mathbf{V}(V, L) = \bigotimes_{x \in L} V_x^*$, with its usual scheme structure.

In fact, the Segre example is just the first in a family. Let $d \geq 0$ be an integer. As a vector space, the algebra $P(V, L)$ breaks up into isotypic pieces for the action of the group $\prod_{x \in L} \mathrm{GL}(V_x)$; each such piece corresponds to a family of partitions indexed by L. Let $I_d(V, L)$ be the sum of all pieces in which some partition has more than d rows. One easily checks that $I_d(V, L)$ is an ideal

of $P(V,L)$, and so cuts out a closed subscheme $\mathrm{Sub}_d(V,L)$ of $\mathbf{V}(V,L)$. Geometrically, a point v in $\mathbf{V}(V,L)$ belongs to $\mathrm{Sub}_d(V,L)$ if and only if for each $x \in L$ there exists a quotient U_x of V_x of dimension at most d such that v belongs to $\bigotimes_{x \in L} U_x^*$. For further discussion of these subspace varieties, see [LW2, §3].

Now, it is clear that $\mathrm{Sub}_d(V,L)$ is functorial for maps in Vec^f. However, for $d \geq 2$, it is not functorial for maps in Vec^Δ, and is therefore not a Δ-scheme. Nonetheless, Sub_d contains a unique maximal Δ-scheme, which we call $\Delta\mathrm{Sub}_d$. Geometrically, we have

$$\Delta\mathrm{Sub}_d(V,L) = \bigcap \mathrm{Sub}_d(V, \mathscr{U}),$$

where the scheme-theoretic intersection is taken over all partitions $\mathscr{U}$ of L. Algebraically, $\Delta\mathrm{Sub}_d$ corresponds to the Δ-ideal generated by the $I_d(V,L)$. It follows from [LW2, Thm 3.1] that the Δ-ideal corresponding to $\Delta\mathrm{Sub}_{d-1}$ (for $d > 1$) is generated by any element of the one dimensional space

$$\textstyle\bigwedge^d \mathbf{C}^d \otimes \bigwedge^d \mathbf{C}^d \subset \mathrm{Sym}^d(\mathbf{C}^d \otimes \mathbf{C}^d) \subset P_2(\mathbf{C}^d, \mathbf{C}^d).$$

One can describe $\Delta\mathrm{Sub}_d$ as the largest Δ-scheme whose coordinate ring has the property that every partition appearing in it has at most d rows.

The Δ-scheme $\Delta\mathrm{Sub}_0$ is just the initial Δ-scheme, i.e., the origin in $\mathbf{V}$. The Δ-scheme $\Delta\mathrm{Sub}_1$ coincides with Sub_1, and is the Segre embedding. The Δ-scheme $\Delta\mathrm{Sub}_2$ (which does not coincide with Sub_2) is equal to the secant variety of the Segre embedding — this assertion is essentially equivalent to the recently proved GSS conjecture (see [Ra]). We do not know an elegant geometric description of $\Delta\mathrm{Sub}_d$ for $d > 2$.

The category of Δ-schemes is closed under several natural operations. Let X and Y be two Δ-schemes. Then the union $X \cup Y$ and scheme-theoretic intersection $X \cap Y$ of X and Y inside of $\mathbf{V}$ are Δ-schemes. In this way, the poset of Δ-schemes is a lattice. Let $X + Y$ be the scheme-theoretic image of the map $X \times Y \to \mathbf{V}$ given by $(x, y) \mapsto x + y$. Then $X + Y$ is a Δ-scheme. In particular, the secant schemes to a Δ-scheme are again Δ-schemes. The same holds for tangent schemes. The functor X_{red} which attaches to (V,L) the reduced subscheme $X_{\mathrm{red}}(V,L)$ of $X(V,L)$ is a Δ-scheme. Starting with the $\Delta\mathrm{Sub}_d$ and using these operations, one obtains a large number of examples of Δ-schemes.

4.3. **Syzygies of Δ-schemes.** Fix a Δ-scheme X. Let $R(V,L)$ be the affine coordinate ring of $X(V,L)$, a quotient of the polynomial ring $P(V,L)$. Put

$$F_p(V,L) = \mathrm{Tor}_p^{P(V,L)}(R(V,L), \mathbf{C}).$$

As discussed in the introduction, the $F_p(V,L)$ record the syzygies of $R(V,L)$ as a $P(V,L)$-module. The functorial properties of R, P and Tor imply that F_p is a Δ-module; this will be made more clear in the proof of the following theorem. Let $F_p^{(d)}$ be its degree d piece; it too is a Δ-module. Our main result on syzygies is the following theorem:

Theorem 4.1. *The Δ-module $F_p^{(d)}$ is small.*

Proof. Let $W(V,L) = \bigotimes_{x \in L} V_x$ be the degree one piece of $P(V,L)$. The $P(V,L)$-module $\mathbf{C}$ admits a Koszul resolution, the terms of which are $P(V,L) \otimes \bigwedge^i W(V,L)$. Tensoring this over $P(V,L)$ with $R(V,L)$, we find that there is a complex computing $F_p(V,L)$ whose terms are $R(V,L) \otimes \bigwedge^i W(V,L)$. Put

$$M_{i,j}(V,L) = R^{(i)}(V,L) \otimes \textstyle\bigwedge^j W(V,L).$$

We then find that $F_p^{(d)}(V,L)$ is the homology of a complex

$$M_{d-p-1,p+1}(V,L) \to M_{d-p,p}(V,L) \to M_{d+1-p,p-1}(V,L)$$

Now, the functor $(V, L) \mapsto M_{i,j}(V, L)$ is easily seen to be a Δ-functor. A short calculation with the Koszul complex shows that the differentials in the above complex respect the Δ-module structure on the $M_{i,j}$. Thus $F_p^{(d)}$, as a Δ-module, is the homology of the complex of Δ-modules

$$M_{d-p-1,p+1} \to M_{d-p,p} \to M_{d+1-p,p-1}.$$

We now claim that $M_{i,j}$ is a small Δ-functor. Indeed, $R^{(i)}$ is a quotient of $P^{(i)} = \Phi(\mathrm{Sym}^i)$, which is a quotient of $\Phi(T_i)$. Of course, $(V, L) \mapsto \bigwedge^j W(V, L)$ is the Δ-module $\Phi(\bigwedge^j)$, and thus a quotient of $\Phi(T_j)$. Thus $M_{i,j} = R^{(i)} \boxtimes \Phi(\bigwedge^j)$ a quotient of $\Phi(T_i) \boxtimes \Phi(T_j) = \Phi(T_{i+j})$, and is thus small. Now, a subquotient of a small Δ-module is small; thus $F_p^{(d)}$ is small. This completes the proof. $\qquad\square$

Corollary 4.2. *The Δ-module $F_p^{(d)}$ is finitely generated.*

Thus there exists a finite list of p-syzygies of degree d for the schemes $X(V, L)$ which give rise to all p-syzygies of degree d under the basic operations on the Δ-module $F_p^{(d)}$. In view of this corollary, the statement "F_p is finitely generated as a Δ-module" is equivalent to the statement "$F_p^{(d)} = 0$ for $d \gg 0$." This probably does not hold in full generality, but in §4.5 we single out a large class of Δ-schemes for which it might hold.

Corollary 4.3. *The series $[F_p^{(d)}]^*$ is a rational function in the s_λ.*

This essentially shows that the information content of the p-syzygies of degree d of X is finite. As we have said, this series can be computed algorithmically (this is discussed below). Thus, essentially all the information about p-syzygies of degree d of X can be computed in finite time!

4.4. **Computational aspects.** We now elaborate on the remark we have made that our proofs give algorithms to calculate the relevant objects. Keep the same notation as the previous section. Say we would like to compute generators for the Δ-module $F_p^{(d)}$ of p-syzygies of degree d. (The algorithm for computing the rational function $[F_p^{(d)}]^*$ proceeds along similar lines.) We proceed as follows. First, $F_p^{(d)}$ is the homology of the sequence

$$M_{d-p-1,p+1} \to M_{d-p,p} \to M_{d+1-p,p-1}.$$

As explained, $R^{(i)}$ is a quotient of $\Phi(T_i)$ while $(V, L) \mapsto \bigwedge^j W(V, L)$ is a quotient of $\Phi(T_j)$, and so $M_{i,j}$ is a quotient of $\Phi(T_{i+j})$. In particular, each module in the above complex is a quotient of $\Phi(T_d)$. This shows that each is canonically a $W_d^{S_d}$-module and the differentials respect this structure. Now, W_d only has partitions with at most d rows; the same is true for the $M_{i,j}$ above. Thus it suffices to see what happens when we evaluate on $\mathbf{C}^d$. Precisely, let A be the twisted commutative algebra $L \mapsto W_d^{S_d}((\mathbf{C}^d)_L)$ and let $N_{i,j}$ be the A-module given by $L \mapsto M_{i,j}((\mathbf{C}^d)_L)$, where $(\mathbf{C}^d)_L$ denotes the constant family on $\mathbf{C}^d$. Let E be the homology of the complex

$$N_{d-p-1,p+1} \to N_{d-p,p} \to N_{d+1-p,p-1};$$

thus $E_L = F_p^{(d)}((\mathbf{C}^k)_L)$. The row bounds then imply that generators for E as an A-module are generators for $F_p^{(d)}$ as a $W_d^{S_d}$-module. Proposition 2.9 thus implies that generators for E as an A-module are generators for $F_p^{(d)}$ as a Δ-module.

Now, A is equal to $\mathrm{Sym}(U\langle 1 \rangle)^{S_d}$ where $U = (\mathbf{C}^d)^{\otimes d}$. It follows that any partition in A has at most $\dim U = d^d$ rows. Since $M_{d-i,i}$ is a subquotient of W_d, it follows that $N_{d-i,i}$ is a subquotient of $\mathrm{Sym}(U\langle 1 \rangle)$. Thus any partition appearing in $N_{d-i,i}$ has at most d^d rows as well. Thus we do not loose information by evaluating on $\mathbf{C}^{d^d}$ (regarding everything in the Schur model). That is, generators for $E(\mathbf{C}^{d^d})$ as an $A(\mathbf{C}^{d^d})$-module give generators for E.

Finally, $A(\mathbf{C}^{d^d})$ is the subring of the polynomial ring in d^{2d} variables which are S_d-invariant. Each of the modules $N_{d-i,i}(\mathbf{C}^{d^d})$ is a finite module over this ring. And $E(\mathbf{C}^{d^d})$ is the homology of the

complex $N_\bullet$ at $i = p - 1$. We have thus reduced the problem to a computation involving explicitly described finitely generated rings and modules. These computations can be done algorithmically.

4.5. Δ-schemes of finite level.
A Δ-scheme X (resp. Δ-ideal I) has *level* $\leq d$ if every partition appearing in its coordinate ring has at most d rows. One easily sees that X has level $\leq d$ if and only if it is contained in $\Delta\mathrm{Sub}_d$. We say that X has *finite level* if it has level $\leq d$ for some d. All the operations we have discussed (union, intersection, sum, formation of secant, tangent and reduced schemes) preserve the finite level condition. We thus have many examples of Δ-schemes of finite level. Note, however, that the final Δ-scheme $\mathbf{V}$ does not have finite level.

The following three statements are equivalent:

(a) Every ascending chain of Δ-ideals of finite level stabilizes.
(b) Every Δ-ideal of finite level is finitely generated.
(c) Every Δ-ideal I of finite level is "uniformly generated," i.e., there exists an integer d such that $I(V, L)$ is generated in degrees $\leq d$ for all (V, L).

It is clear that (a) and (b) are equivalent, and clear that each implies (c). That (c) implies (b) follows from Corollary 4.2 and the discussion following it.

It seems reasonable to hope that statements (a)–(c) are in fact true. Indeed, there has been some recent work establishing that particular finite level Δ-ideals are finitely generated:

- Raicu [Ra] has proved the GSS conjecture, which implies that the ideal of the first secant variety to the Segre embedding is finitely generated as a Δ-ideal.
- Draisma and Kuttler [DK] have shown that the secant varieties to the Segre varieties are all cut out set-theoretically by equations of a bounded degree. (In fact, in [DK, §7], a conjecture concerning something similar to statements (a)–(c) above is posed.)
- Landsberg and Weyman [LW] have obtained results about the ideal of the tangent variety to the Segre, which imply that it is finitely generated as a Δ-ideal.

A proof of the statements (a)–(c) would have great consequences: for instance, it would essentially encompass all of the above results. Assuming these statements are true, it would be reasonable to believe that the pth syzygy module of a finite level Δ-scheme is supported in finitely many degrees.

5. Application to Segre varieties

We now apply the theory we have developed to the study of syzygies of Segre embeddings.

5.1. Theorems A and B.
We begin by establishing the theorems stated in the introduction. Let F_p be the Δ-module of p-syzygies of the Segre embedding, as defined in §4.3. By Theorem 4.1 and its corollaries, Theorems A and B follow from the following result, which states that F_p has only finitely many non-zero graded pieces. This result is well-known to the experts, so we only give a brief proof. We thank Aldo Conca for showing it to us.

Proposition 5.1. *For $p \geq 1$, the space F_p is supported in degrees $p + 1, \ldots, 2p$.*

Proof. This follows from the fact that the ideal of the Segre variety is generated by a Gröbner basis of degree 2 [ERT, Prop. 17], together with general facts about Gröbner bases (such as the Taylor resolution, see [E, Exercise 17.11]). □

Remark 5.2. The bound in the proposition is not optimal. Indeed, it is known [Ru] that F_2 and F_3 are supported in degrees 3 and 4, while the upper bounds provided by the proposition are 4 and 6. The optimal upper bound is not known. However, our proof of Theorem A provides an algorithm for finding it for any particular value of p.

5.2. **The work of Lascoux.** Lascoux determined the entire minimal resolution of certain determinantal varieties (see [L], and also [PW], where a gap in [L] is resolved). The rank 1 case of his result exactly gives the leading term of our series f_p. We recall his results in our language.

First, we discuss some terminology. Let $m = s_{\lambda_1} \cdots s_{\lambda_n}$ be a monomial in the variables s_λ. We say that m has *order n*. We say that m has *degree d* if each λ_i is a partition of d. Every term in the series $[F_p^{(d)}]$ has degree d, while the orders of the terms are unbounded. The degree d, order n terms in f_p give information about the p-syzygies of degree d for the Segre embedding of an n-fold product of projective spaces.

Let $f_{p,2}$ be the order two two term of f_p. This is the leading order term of f_p. We consider its degree d piece $f_{p,2}^{(d)}$. Write $d = p + h$. Of course, if $h \leq 0$ then $f_{p,2}^{(d)} = 0$. Proposition 5.1 implies that $f_{p,2}^{(d)} = 0$ for $h > p$. Lascoux gives a much better bound: $f_{p,2}^{(d)} = 0$ for $h > \sqrt{p}$. Assume now $1 \leq h \leq \sqrt{p}$. Let S be the set of pairs of partitions (α, β) such that α has at most h columns, β has at most h rows and $|\alpha| + |\beta| = p - h^2$. Associate to (α, β) a new pair of partitions (μ, ν) as follows. Start with a rectangle with h columns and $h + 1$ rows. To get μ, append α to the bottom and β to the right. To get ν, append the dual of β to the bottom and the dual of α to the right. Lascoux's result is then

$$f_{p,2}^{(d)} = \tfrac{1}{2} \sum_{(\alpha,\beta)\in S} s_\mu s_\nu.$$

For example, say $p = 1$ and $d = 2$. Then $h = 1$. Since $p - h^2 = 0$ the set S consists of the single pair (α, β) where $\alpha = \beta$ is the zero partition. The partitions μ and ν are both $(1, 1)$ and so we find $f_{1,2}^{(2)} = \tfrac{1}{2} s_{(1,1)}^2$, i.e., the quadratic piece of the ideal of the embedding of $\mathbf{P}(V_1) \times \mathbf{P}(V_2)$ is $\bigwedge^2 V_1 \otimes \bigwedge^2 V_2$.

5.3. **The polynomial g_p.** Let $G_p = \Psi(F_p)$ be the cokernel of the map $\Delta F_p \to F_p$. Since F_p is a finitely generated Δ-module, G_p is a finite length object of the category $\mathrm{Sym}(\mathcal{S})$. The object G_p records precisely those syzygies that cannot be built out of syzygies on a product of fewer projective spaces. We let g_p be the *polynomial* $[G_p]$ (and $g_p^* = [G_p]^*$). The objects $L^i \Psi F_p$ for $i \geq 1$ are important as well — indeed, $[F_p]$ can be recovered from them — though they are bit less accessible.

We remark that our two main theorems can be rephrased using f_p and g_p so as to look more similar: Theorem A is exactly the statement that g_p^* is a polynomial, while Theorem B is exactly the statement that f_p^* is a rational function.

5.4. **An Euler characteristic.** Define

$$\chi = \sum_{p \geq 0} (-1)^p f_p.$$

There are only finitely many terms of a given degree and order in the sum, and so it makes sense. We remark that $f_0 = 1$ — the first term in the resolution of R is always $P = P \otimes \mathbf{C}$ and so $F_0 = \mathbf{C}$ for any (V, L). The main result of this section is an explicit computation of χ. The notation used in the following proposition is defined below it.

Proposition 5.3. *We have*

$$\chi = \left[\sum_{k=0}^\infty \exp(s_{(k)}) \right] \boxtimes \left[\sum_\lambda \kappa_\lambda \exp(s_\lambda') \right],$$

where κ_λ is the rational number $(-1)^{|\lambda|} \mathrm{sgn}(c_\lambda) z_\lambda^{-1}$. The second sum is taken over all partitions λ — including $\lambda = 0$, where the term is 1.

Extracting the degree k piece of the above formula yields:

Corollary 5.4. *For $k > 0$ we have*

$$\chi^{(k)} = \sum_{p=0}^{k} \left[\frac{(-1)^p}{p!} \sum_{\lambda \vdash p} (\#c_\lambda) \operatorname{sgn}(c_\lambda) \exp(s_{(k-p)} \boxtimes s'_\lambda) \right].$$

The $p = 0$ term of the above sum is $\exp(s_{(k)})$. We have $\chi^{(0)} = 1$.

We now define notation that will be in place for the rest of the section (and is used in the above proposition). Let λ be a partition of p. We let c_λ denote the conjugacy class of S_p corresponding to λ, normalized so that $\lambda = (1, \ldots, 1)$ corresponds to the identity element, and we let $z_\lambda = p!/\#c_\lambda$ be the order of the centralizer of any element of c_λ. We let χ_λ denote the character of S_p corresponding to λ, normalized so that $\lambda = (1, \ldots, 1)$ corresponds to the sign character sgn. The notation s_λ means what is has meant previously, namely the element $[\mathbf{S}_\lambda]$ of $K(\mathcal{S})$; in particular, $s_{(k)} = [\operatorname{Sym}^k]$. We define s'_λ to be the element of $K(\mathcal{S})$ of degree p which corresponds to the class function on S_p supported on c_λ and taking value z_λ there. Explicitly,

$$s'_\lambda = \sum_{\mu \vdash p} \chi_\mu(c_\lambda) s_\mu.$$

The symbol $\boxtimes$ is the point-wise tensor product: $s_\lambda \boxtimes s_\mu$ is computed using the Littlewood–Richardson rule. As usual, $W = W_1$ is the object of $\operatorname{Sym}(\mathcal{S})$ which assigns to (V, L) the tensor product of the V's. Throughout this section $\bigwedge^i W$ and $\mathbf{S}_\lambda(W)$ refer to point-wise operations in $\operatorname{Sym}(\mathcal{S})$. For instance, $\bigwedge^i W$ is the object of $\operatorname{Sym}(\mathcal{S})$ given by

$$(\textstyle\bigwedge^i W)(V, L) = \textstyle\bigwedge^i (W(V, L)) = \bigwedge^i \left(\bigotimes_{x \in L} V_x \right).$$

We now begin proving Proposition 5.3. We begin with the following.

Lemma 5.5. *We have*

$$\chi = [R] \boxtimes \left(\sum_{p=0}^{\infty} (-1)^p [\textstyle\bigwedge^p W] \right).$$

Proof. As discussed in the proof of Theorem 4.1, F_p is the homology of the complex $R \boxtimes \bigwedge^i W$ at $i = p - 1$. The formula follows from standard facts about Euler characteristics. $\qquad\square$

Lemma 5.6. *We have $[R^{(d)}] = \exp(s_{(d)})$, and so $[R] = \sum_{d \geq 0} \exp(s_{(d)})$.*

Proof. We have

$$R_n^{(d)}(V_1, \ldots, V_n) = \operatorname{Sym}^d(V_1) \otimes \cdots \otimes \operatorname{Sym}^d(V_n).$$

Thus $[R_n^{(d)}]$ is equal to $\frac{1}{n!}[\operatorname{Sym}^d]^n$ in $\operatorname{Sym}^n(K(\mathcal{S}))$. The result follows. $\qquad\square$

Lemma 5.7. *Let λ be a partition of $p > 0$. Let $s_{\lambda,n}$ be the class in $\operatorname{Sym}^n(K(\mathcal{S}))$ of the functor $(V_1, \ldots, V_n) \mapsto \mathbf{S}_\lambda(V_1 \otimes \cdots \otimes V_n)$. Then*

$$\sum_{n=0}^{\infty} s_{\lambda,n} = \frac{1}{p!} \sum_{\mu \vdash p} (\#c_\mu) \chi_\lambda(c_\mu) \exp(s'_\mu).$$

Note that the left side above is nothing other than $[\mathbf{S}_\lambda(W)]$.

Proof. A simple manipulation shows that for any vector spaces U and V we have

$$\mathbf{S}_\lambda(U \otimes V) = \bigoplus C_{\lambda\mu\nu} \mathbf{S}_\mu(U) \otimes \mathbf{S}_\nu(V),$$

where the sum is over all partitions μ and ν of p, and

$$C_{\lambda\mu\nu} = \dim(\mathbf{M}_\lambda \otimes \mathbf{M}_\mu \otimes \mathbf{M}_\nu)^{S_p} = \dim \operatorname{Hom}_{S_p}(\mathbf{M}_\lambda, \mathbf{M}_\mu \otimes \mathbf{M}_\nu).$$

(This appears as Exercise 6.11(b) in [FH].) We therefore have

$$\mathbf{S}_\lambda(V_1 \otimes \cdots \otimes V_n) = \bigoplus_{\mu,\nu} C_{\lambda\mu\nu}\mathbf{S}_\mu(V_1) \otimes \mathbf{S}_\nu(V_2 \otimes \cdots \otimes V_n).$$

We thus have a recurrence

$$s_{\lambda,n} = \frac{1}{n}\sum_{\mu,\nu} C_{\lambda\mu\nu}s_\mu s_{\nu,n-1}.$$

It will now be convenient to switch from working in the degree p piece of $K(\mathcal{S})$ to working in $K(S_p)$. The two are in isomorphism via $s_\lambda = [\mathbf{S}_\lambda] \leftrightarrow [\mathbf{M}_\lambda]$. Thus $s_{\lambda,n}$ can be regarded as an element of $\mathrm{Sym}^n(K(S_p))$. Note that the sum $\sum_\mu C_{\lambda\mu\nu}s_\mu$ is equal to $[\mathbf{M}_\lambda \otimes \mathbf{M}_\nu]$. We can thus rephrase our last expression as follows. Let v_n be the column vector $(s_{\lambda,n})_\lambda$ and let A be the matrix $([\mathbf{M}_\lambda \otimes \mathbf{M}_\mu])_{\lambda,\mu}$. Then

$$v_n = \tfrac{1}{n}Av_{n-1}.$$

We thus have

$$\sum v_n = \exp(A)v_0.$$

Note that v_0 has a 1 in the entry $\lambda = (p)$ and a 0 in all other entries. Indeed, an empty tensor product is equal to $\mathbf{C}$, so $s_{\lambda,0}$ is the class of $S_\lambda(\mathbf{C})$ in $\mathrm{Sym}^0(K(S_p)) = \mathbf{Q}$; in other words, $a_{\lambda,0} = \dim\mathbf{S}_\lambda(\mathbf{C})$. This is 1 if $\lambda = (p)$ and 0 otherwise. We therefore find that the initial vector v_0 in the above recurrence is quite simple. The problem is to determine the exponential of the matrix A. We will achieve this by diagonalizing A.

Let B be the matrix $(\chi_\lambda(c_\mu))_{\lambda,\mu}$. We index by rows first, then columns. Thus the rows of B are indexed by irreducible characters and the columns by conjugacy classes; B is the character table of S_p. Let D be the diagonal matrix given by $D_{\lambda\lambda} = z_\lambda\delta_\lambda$ where $z_\lambda = p!/\#c_\lambda$ is the cardinality of the centralizer of c_λ and δ_λ is the class function on S_p which assigns c_λ the value 1 and all other conjugacy classes 0. We then have the following fundamental identity

(6) $$AB = BD.$$

We now explain this identity. First, we regard the entries of A as class functions, so $A_{\lambda\mu}$ is the character of $\mathbf{M}_\lambda \otimes \mathbf{M}_\mu$. The entry of the product AB at (λ,μ) evaluated at c_ζ is thus given by

$$\left(\sum_\nu A_{\lambda\nu}B_{\nu\mu}\right)(c_\zeta) = \sum_\nu \chi_\lambda(c_\zeta)\chi_\nu(c_\zeta)\chi_\nu(c_\mu) = \chi_\lambda(c_\zeta)\sum_\nu \chi_\nu(c_\zeta)\chi_\nu(c_\mu).$$

Now, for g and h in S_p the sum $\sum\chi_\nu(g)\chi_\nu(h)$ is the trace of (g,h) acting on the representation $\mathbf{C}[S_p]$ of $S_p \times S_p$. Since this is a permutation representation, the trace is given by the number of fixed points. An element $x \in S_p$ is a fixed point if $gxh^{-1} = x$, or equivalently, if $g = xhx^{-1}$. Thus the number of fixed points is 0 if g and h are not conjugate, and is otherwise the size of the centralizer of g. We therefore find

$$\chi_\lambda(c_\zeta)\sum_\nu \chi_\nu(c_\zeta)\chi_\nu(c_\mu) = \chi_\lambda(c_\mu)z_\mu\delta_\mu(c_\zeta) = B_{\lambda\mu}D_{\mu\mu}(c_\zeta).$$

This proves (6).

The equation (6) diagonalizes A. However, for it to be useful we need to compute B^{-1}. This is straightforward. Let C be the diagonal matrix given by $C_{\lambda\lambda} = z_\lambda^{-1}$. Then the orthonormality of characters is precisely the identity

$$BCB^t = 1$$

and so $B^{-1} = CB^t$. (This again uses the fact that all representations of symmetric groups are self-dual, which is equivalent to their characters being real valued.)

We now find

$$\exp(A)v_0 = B\exp(B^{-1}AB)B^{-1}v_0 = B\exp(D)CB^t v_0.$$

Simple matrix multiplication now gives the stated formula. $\square$

$s = [\mathrm{Sym}^2], \quad w = [\bigwedge^2]$
$f_1 = \frac{1}{2}e^{s+w} + \frac{1}{2}e^{s-w} - e^s$
$g_1 = \frac{1}{2}w^2$

$s = [\mathrm{Sym}^3], \quad w = [\bigwedge^3], \quad t = [\mathbf{S}_{(2,1)}]$
$f_2 = \frac{1}{3}e^{s+w+2t} - \frac{1}{3}e^{s+w-t} - e^{s+t} + e^s$
$g_2 = wt$

$s = [\mathrm{Sym}^4], \quad w = [\bigwedge^4], \quad a = [\mathbf{S}_{(3,1)}], \quad b = [\mathbf{S}_{(2,2)}], \quad c = [\mathbf{S}_{(2,1,1)}]$
$f_3 = \frac{1}{8}e^{s+w+3a+2b+3c} - \frac{1}{8}e^{s+w-a+2b-c} + \frac{1}{4}e^{s-w-a+c} - \frac{1}{4}e^{s-w+a-c}$ $\quad + \frac{1}{2}e^{s+b-c} - \frac{1}{2}e^{s+2a+b+c} + e^{s+a} - e^s.$
$g_3 = aw + \frac{1}{2}c^2 \quad (+?)$

FIGURE 1. Values of f_p and g_p for p small.

Lemma 5.8. *Let x and y belong to $K(\mathcal{S})$. Then $\exp(x) \boxtimes \exp(y) = \exp(x \boxtimes y)$.*

Proof. For an object F of $\mathcal{S}$ let F' be the object of $\mathrm{Sym}(\mathcal{S})$ given by $(V, L) \mapsto \bigotimes_{x \in L} F(V_x)$. Then $\exp([F]) = [F']$. Now, let x and y in $K(\mathcal{S})$ be given. Since $\boxtimes$ is additive and $\exp$ is multiplicative, it suffices to treat the case where $x = [F]$ and $y = [G]$, with F and G in $\mathcal{S}$. We then have

$$\exp(x) \boxtimes \exp(y) = [F'] \boxtimes [G'] = [F' \boxtimes G'] = [(F \boxtimes G)'] = \exp([F \boxtimes G]) = \exp(x \boxtimes y).$$

The key fact is the obvious formula $F' \boxtimes G' = (F \boxtimes G)'$.   □

The proposition and corollary follow easily from the above lemmas.

5.5. **Examples for small p.** The main result of [Ru] states that Segre embeddings satisfy the Green–Lazarsfeld property N_3 but not N_4. This means that F_1, F_2 and F_3 are supported exactly in degrees 2, 3 and 4 respectively (and that F_4 has support outside degree 5). From this, we deduce the following equalities:

$$f_1 = -\chi^{(2)}, \qquad f_2 = \chi^{(3)}, \qquad f_3 = -\chi^{(4)}, \qquad f_4^{(5)} = \chi^{(5)}.$$

These values have been computed in Proposition 5.3. They are listed explicitly, and in simplified form, in Figure 1 (other than $f_4^{(5)}$). We explain how the value for f_1 given in the figure was derived, the values of f_2 and f_3 being gotten in a similar fashion. Proposition 5.3 gives

$$f_1 = -\exp(s_{(2)}) + \exp(s_{(1)} \boxtimes s'_{(1)}) - \frac{1}{2}\left(\exp(s'_{(1,1)}) - \exp(s'_{(2)})\right).$$

We have $s'_{(1)} = s_{(1)}$, while $s'_{(1,1)} = s_{(2)} + s_{(1,1)}$ and $s'_{(2)} = s_{(2)} - s_{(1,1)}$. Now, the product $s_{(1)} \boxtimes s_{(1)}$ is just the usual product in $K(\mathcal{S})$, i.e., it is the class of the functor $V \mapsto \mathrm{Sym}^1(V) \otimes \mathrm{Sym}^1(V)$. This, of course, is equal to $s_{(2)} + s_{(1,1)}$. We thus find

$$f_1 = \frac{1}{2}\exp(s_{(2)} + s_{(1,1)}) + \frac{1}{2}\exp(s_{(2)} - s_{(1,1)}) - \exp(s_{(2)}).$$

This is the value given in the figure.

We have previously stated (without proof) that the equation for $\mathbf{P}^1 \times \mathbf{P}^1$ in $\mathbf{P}^3$ generates F_1 as a Δ-functor. This implies that g_1 is the order two piece of f_1. Similarly, we have stated (without proof) that any non-zero syzygy for $\mathbf{P}^1 \times \mathbf{P}^2$ in $\mathbf{P}^5$ generates F_2. This implies that g_2 is the order two piece of f_2. Finally, the order two piece of g_3 is the same as that of f_3; we are unaware if g_3 has any terms of higher order. These remarks explain the values of g_1, g_2 and g_3 given in the figure.

6. Questions and problems

(**1**) Are finitely generated twisted commutative algebras noetherian? We proved this for algebras generated in order 1, and can also prove it for certain algebras in order 2.

(**2**) Are finite level Δ-ideals finitely generated? More generally, are the Δ-modules F_p for a finite level Δ-scheme concentrated in finitely many degrees?

(**3**) Let M be a finitely generated module over a twisted commutative algebra finitely generated in order 1. We have shown that $H_M(t)$ is a polynomial of t and e^t. Our proof shows that the maximal power of e^t is related to the number of rows in M. How else does the form of $H_M(t)$ relate to the structure of M?

(**4**) Our series f_p forgets the S_n-equivariance on $F_{p,n}$. Can one modify f_p to retain this information, and still have something resembling a rationality result?

(**5**) Is the series f_p a polynomial in the s_λ and the $e^{\pm s_\lambda}$? This does not follow from what we have proved, but one might hope that it is true based on some of our results. In fact, based on the computations of f_1, f_2 and f_3 for the Segre, one might hope for a stronger statement: f_p is a polynomial in only the $e^{\pm s_\lambda}$. We suspect this is false, but do not know.

(**6**) Compute f_p and g_p for more values of p or for Δ-schemes other than the Segre. We have given an algorithm to do this, but it is too inefficient to use. It would be particularly interesting to compute f_4 for the Segre since this is the first place where the Green–Lazarsfeld property fails and the value is not given by the Euler characteristic formula.

(**7**) Is the series $\sum_{i \geq 0} (-1)^i [L^i \Psi F_p]^* q^i$ a rational function of q and the s_λ? This series contains more information than f_p and g_p, since one can recover f_p^* by applying Φ and setting $q = 1$, and one can recover g_p^* by setting $q = 0$.

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